Type II Differential Topology

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Abstract

In this lecture we showcase an early application of noncommutative geometry — the proof of the Novikov Conjecture for Gromov-hyperbolic groups. (Connes-Moscovici, *Cyclic cohomology, the Novikov conjecture, and hyperbolic groups*, Topology **29**(1990), 345–388).

The index theorem

Theorem 1. (Atiyah-Singer) *Let D be the Dirac operator on a compact even-dimensional spin manifold M; then* $\mathrm{Index}(D) = \int_M \widehat{\mathcal{A}}(M)$.

Heat equation proof. Let K_t^+ and K_t^- be the $\pm$-graded parts of the Heat operator e^{-tD^2}. Then

$$\mathrm{Index}(D) = \mathrm{Tr}\, K_t^+ - \mathrm{Tr}\, K_t^-$$

for all values of time t. On the other hand as $t \downarrow 0$ the heat kernels $K_t^\pm$ localize near the diagonal in $M \times M$ and there are asymptotic expansions of the form

$$\mathrm{tr}\, K_t(x, x) \sim a_m(x) t^{-m} + a_{m-1}(x) t^{-m+1} + \cdots$$

where $m = \frac{1}{2} \dim M$. The existence of this asymptotic expansion is what is responsible for the analytic continuation of the ζ function.

Atiyah-Patodi-Singer identified the difference of the terms $a_0^+ - a_0^-$ with the $\widehat{\mathcal{A}}$ form. The proof was later simplified by Getzler using a Clifford symbolic calculus for pseudodifferential operators on spinors.

Local index theorem

One can regard the heat equation formula as follows: the index of D defines an element of $K_0(\mathcal{R})$, where $\mathcal{R}$ is the algebra of smoothing operators; and we are computing the index by pairing with $[\mathrm{Tr}] \in HC^0(\mathcal{R})$. Are there other interesting cyclic cocyles for $\mathcal{R}$?

Answer: No. ($\mathcal{R}$ is essentially a big matrix algebra.)

However, suppose that we pass to the sub 'algebra' $\mathcal{R}_\epsilon$ of operators of propagation $< \epsilon$. Every Alexander-Spanier cocycle $c(x_0, \ldots, x_q)$ on M defines a cyclic cocyle τ_c on $\mathcal{R}_\epsilon$ by sending $(K^0, \ldots, K^q)$ to

$$\iint c(x_0, \ldots, x_q) K^0(x_0, x_1) K^1(x_1, x_2) \cdots K^q(x_q, x_0).$$

Theorem 2. (Connes-Moscovici) *One has*

$$\tau_{c*}(\mathrm{Index}\, D) = \mathsf{const} \cdot \int_M \widehat{\mathcal{A}}(M) \smile c.$$

Proof is an application of the Getzler calculus.

Elliptic families

Let B be a space and let $E \to B$ be a fiber bundle with manifold fibers. A family of elliptic operators along the fibers has an index which belongs to $K^0(B)$.

Example (Lusztig) Let M be a torus, $\mathbb{R}^n/\mathbb{Z}^n$, n even. The characters χ of $\pi_1 M$ form a group B (which is also a torus!) Every such character corresponds to a flat line bundle. Let D be the signature operator of M; then for each character $\chi \in B$ we may form the twisted signature operator D_χ, and in this way we obtain a family $\tilde{D}$ of elliptic operators parameterized by χ. Thus we get a 'higher signature' $\mathrm{Index}(\tilde{D}) \in K^0(B)$.

Theorem 3. *The higher signature* $\mathrm{Index}(\tilde{D})$ *is an invariant of the oriented homotopy type of* M*.*

Compare the homotopy invariance of the ordinary signature. . .

Noncommutative families

For a different view of Lusztig's construction, consider the lift $\tilde{D}$ of D to the universal covering space $\tilde{M}$ of M. Let $\Gamma = \pi_1(M)$ ($= \mathbb{Z}^m$ in this example.) Let $\widetilde{\mathcal{R}}(M)$ be the algebra of smoothing operators on $\tilde{M}$ which are *of finite propagation* and also Γ-*invariant* (this is the groupoid algebra of the fundamental groupoid of M). Then $\tilde{D}$ is invertible modulo $\widetilde{\mathcal{R}}(M)$. Consequently there is defined an index $\mathrm{Index}(\tilde{D}) \in K_0(\widetilde{\mathcal{R}}(M))$.

Lemma 1. *The C^*-algebra completion of $\widetilde{\mathcal{R}}(M)$, in the norm topology of $\mathfrak{B}(L^2(\tilde{M}))$, is Morita equivalent to $C_r^*(\Gamma)$.*

In the case of the torus, $C_r^*(\Gamma) = C(B)$ and the index of $\tilde{D}$ defined by operator algebras equals the higher index in $K^0(B)$ defined on the previous slide.

Theorem 4. *The higher signature $\mathrm{Index}(\tilde{D}) \in K_0(C_r^*(\Gamma))$ is an invariant of the oriented homotopy type of M for every fundamental group Γ.*

The Novikov Conjecture

- Belongs to the realm of *surgery theory*, in which the fundamental question is: What is a manifold?

- Conjecture (and its relatives) reduce the classification of manifolds with given fundamental group Γ to a question of relative homology.

The Novikov Conjecture is this: Let M be a compact manifold with fundamental group Γ and let $c \in H^k(\Gamma; \mathbb{R})$ be a group cohomology class. Then the *Novikov higher signature*

$$\langle L(M) \smile f^*(c), [M] \rangle, \qquad f \colon M \to B\Gamma$$

is an invariant of oriented homotopy type.

Program of proof. Obtain the Novikov higher signatures from $\mathrm{Index}(\widetilde{D}) \in K_0(C^*_r(\Gamma))$.

Application of the local index theorem

Let $c \in H^k(\Gamma; \mathbb{R})$. Then $f^*(c)$ may be represented as an Alexander-Spanier cohomology class on M and therefore defines a cyclic 'cocycle' ϕ_c on $\mathcal{R}_\epsilon(M)$ for sufficiently small ϵ.

Lemma 2. *For sufficiently small ϵ there is a $1:1$ correspondence between $\mathcal{R}_\epsilon(M)$ and $\widetilde{\mathcal{R}}_\epsilon(M)$. Moreover, under this $1:1$ correspondence, ϕ_c corresponds to a genuine cyclic cocyle τ_c on the whole algebra $\widetilde{\mathcal{R}}(M)$.*

Hence

Theorem 5. *For each group cocycle c for M there is a cyclic cocycle τ_c on $\widetilde{R}(M)$ such that $\tau_{c*}(\mathrm{Index}(\tilde{D}))$ is the Novikov higher signature $\langle L(M) \smile f^*(c), [M] \rangle$.*

Almost a proof of Novikov

We have nearly proved the Novikov conjecture.

'All' we need to do is to fill in the dotted arrow in the diagram below

$$\begin{tikzcd}
K_0(\widetilde{\mathcal{R}}(M)) \arrow[r] \arrow[dr, "\tau_{c*}"'] & K_0(C^*_r(\Gamma)) \arrow[d, dotted] \\
 & \mathbb{R}
\end{tikzcd}$$

Problem: The cocycle τ_c need not, a priori, be a k-trace.

Rapid decay

We ask: When is a sequence $\{c_g\}_{g\in\Gamma}$ the sequence of Fourier coefficients of some element of $C_r^*(\Gamma)$?

For groups Γ of polynomial growth it is clear that a sufficient condition is $|c_g| = O(|g|^{-N})$ for N large enough. But such groups are not very interesting.

Haagerup (*An example of a non nuclear C^*-algebra which has the metric approximation property*, Inventiones **50**(1979) 279–293) proved: Let Γ be a free group. Then there is an $N > 0$ such that if the function $g \mapsto |c_g||g|^N$ belongs to $\ell^2(\Gamma)$, then $\sum c_g g$ belongs to $C_r^*(\Gamma)$; in fact he estimated the $C_r^*(\Gamma)$ norm of the sum in terms of the ℓ^2 norm of $g \mapsto |c_g|(1+|g|)^N$.

A group enjoying this property is said to be of *rapid decay* (RD).

Proof of RD

The key lemma in Haagerup's proof of RD for the free group is the following:

Lemma 3. *Let δ and ℓ be given. There exists a constant $C > 0$ such that, for any $g \in \Gamma$, there are at most C ways of decomposing $g = g_1 g_2$ with $|g_1| = \ell$ and $|g_1| + |g_2| \leq |g| + \delta$.*

Exercise: Prove this.

Gromov defined a *hyperbolic* group to be one in which all geodesic triangles are thin. Hyperbolic groups have the large scale qualitative features of free groups. Jolissaint extended Haagerup's calculation to prove

Theorem 6. *Hyperbolic groups enjoy property RD.*

Remark: Not all groups have RD — e.g. no exponential growth solvable group has RD.

Novikov for hyperbolic groups

Lemma 4. (Gromov) *Every cohomology class for a hyperbolic group may be represented by a bounded cocycle.*

Choose such a representative for $c \in H^*(\Gamma; \mathbb{R})$. Then we have

Lemma 5. *The homomorphism* $\tau_{c*}\colon K_0(\widetilde{\mathcal{R}}) \to \mathbb{R}$ *extends to a homomorphism* $K_0(C_r^*(\Gamma)) \to \mathbb{R}$.

The proof uses Haagerup estimates to extend the domain of τ_c to a dense subalgebra of $C_r^*(\Gamma)$ which is smooth (closed under holomorphic calculus). Consequently

Theorem 7. *The Novikov conjecture holds for hyperbolic groups.*

Discussion

This proof is a classic application of cyclic theory to obtain maps which did not (a priori) come from K-homology.

There now exist K-theoretic proofs of NC for hyperbolic groups, using hypereuclideanness, contractibility of the Gromov compactification, amenability of the boundary action, . . . But these seem to use rather different features of hyperbolic geometry — it is an open question to relate the key features of these proofs.

V Lafforgue has shown that uniform lattices in $SL(3, \mathbb{R})$ enjoy RD, and this is a key step in his proof of the Baum-Connes conjecture for such lattices. On the other hand $SL(3, \mathbb{Z})$ does not enjoy RD since it contains an exponential growth solvable subgroup.

Conjecture (Valette) All uniform lattices in $SL(n, \mathbb{R})$ enjoy RD.