Lecture 6
Classification of Factors

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Von Neumann Algebras

$\mathcal{B}(H)$ = Bounded operators on a separable Hilbert space

$\mathcal{L}^1(H)$ = Trace class operators on H

Proposition. *$\mathcal{L}^1(H)^* \cong \mathcal{B}(H)$ via the pairing $\langle T_1, T_2 \rangle = \operatorname{trace}(T_1 T_2)$.*

Definition. The *ultraweak* (u.w.) topology on $\mathcal{B}(H)$ is the associated weak* topology.

Double Commutant Theorem. *A unital $*$-algebra $M \subseteq \mathcal{B}(H)$ is u.w.-closed iff $M'' = M$.*

Definitions.

A $*$-subalgebra $M \subseteq \mathcal{B}(H)$ is a *von Neumann algebra* iff $M = M''$.

M is a factor iff $M \cap M' = \mathbb{C}I$.

Classification into Types

Suppose M is the *commutant* of a unitary representation π (of some object, e.g. a group). Then recall that:

Projections in M $\leftrightarrow$ *Subrepresentations of* π

Moreover there are further correspondences:

Equivalence of projections	$\leftrightarrow$	Equivalence of subrepresentations
Order on equiv alence classes	$\leftrightarrow$	Inclusion of sub-representations

Theorem (Murray and von Neumann). *If* M *is a factor then the ordering on equivalence classes of projections is one of:*

- $\{0, 1, \ldots, n\}$ $\{0, 1, \ldots, \infty\}$ Type I
- $[0, 1]$ $[0, \infty]$ Type II
- $\{0, \infty\}$ Type III

Algebraic versus Spatial Isomporphism

Obvious: Each von Neumann algebra is a dual space.

Not so obvious: The space $M_* \subseteq M^*$ is unique.

$\Rightarrow$ The u.w. topology is intrinsic to M.

Now suppose given $\left.\begin{array}{l} \pi_1 \colon M \to \mathcal{B}(H_1) \\ \pi_2 \colon M \to \mathcal{B}(H_2) \end{array}\right\}$ u.w. continuous

- The commutant of $\pi_1 \oplus \pi_2$ in $\mathcal{B}(H_1 \oplus H_2)$ is a factor (if M is).

- Subreps of $\pi_1 \oplus \pi_2$ (like π_1 and π_2) are classified by dimension of projections:

$$\pi \mapsto \dim(P_\pi) \in \begin{cases} \{0, \ldots, d_{\max}\} \\ [0, d_{\max}] \\ \{0, \infty\} \end{cases}$$

- Hence the representation theory of a given M is essentially trivial. (Especially in the type III case!)

Brauer Theory

K = Field

M = finite-dim'l central simple algebra over K

Theorem. *$M \cong M_j(D)$, for some central division algebra D.*

Definition. $\mathrm{Br}(K)$ = isomorphism classes of central division algebras over K.

Theorem. *$\mathrm{Br}(K)$ is an abelian group with the operation*

$$[D_1] + [D_2] = [D], \qquad D_1 \otimes_K D_2 = M_j(D).$$

For $K = \mathbb{C}$ the group $\mathrm{Br}(K)$ is trivial, but e.g. for $\mathbb{Q}_p$ there is an isomorphism $\mathrm{Br}(K) \cong \mathbb{Q}/\mathbb{Z}$.

Example: For $a, b \in K^\times$ set

$$i^2 = a, \quad j^2 = b, \quad ij = -ji = k.$$

If $\mathrm{char}(K) \neq 2$ then

$$D \cong \begin{cases} M_2(K) & \text{if } aX^2 + bY^2 = 1 \text{ has a sol'n} \\ \text{Div. Ring} & \text{if } aX^2 + bY^2 = 1 \text{ has no sol'n} \end{cases}$$

AFD Factors

Definition. A von Neumann algebra M is *approximately finite-dimensional* if it is generated by an increasing family of f.d. $*$-subalgebras.

Example: R_λ = Powers factor from Lecture 2

= Double commutant of $\otimes M_2(\mathbb{C})$ in a GNS representation

Theorem (Murray and von Neumann). *There is a unique AFD factor of type II_1.*

Compare: Group von Neumann algebras. For non-amenable G, classification is largely unknown.

Proof of Theorem. Uses the trace

$$\begin{cases} \mu_t(E) = \dim(\chi_E(T)) \\ \tau(T) = \int_{\mathrm{Spectrum}(T)} \lambda \, d\mu_T(\lambda) \end{cases}$$

and associated norm $\|S\|_2^2 = \tau(S^*S)$ to compute estimates—does not generalize even to II_∞ (since possibly $\|S\|_2 = \infty$ for all S in approximating family).

ITPFI Factors

$$S_j = \begin{pmatrix} \lambda_1^{(j)} & & \\ & \ddots & \\ & & \lambda_{n_j}^{(j)} \end{pmatrix} \qquad \varphi_j(T) = \mathrm{trace}(S_j T)$$

$$\Lambda = \{\, \lambda_k^{(j)} : 1 \leq k \leq n_j, \quad j = 1, 2, \ldots \}$$

$R_\Lambda =$ Double commutant of $\otimes M_{n_j}(\mathbb{C})$ in GNS representation for $\otimes \varphi_j$

Recall:

Theorem (Powers). *For* $S_j \equiv \begin{pmatrix} \frac{1}{1+\lambda} & \\ & \frac{\lambda}{1+\lambda} \end{pmatrix}$ *and for the values* $0 < \lambda < 1$ *the ITPFI factors obtained are pairwise non-isomorphic.*

A negative indication for the classification problem:

Theorem (Woods). *The moduli space of ITPFI factors is not countably separated.*

Tomita's Modular Theory

- $\varphi\colon M \to \mathbb{C}$, faithful normal state (or weight):

  $\varphi(T^*T) \geq 0$

  $\varphi(T^*T) = 0 \quad \Leftrightarrow \quad T = 0$

  φ u.w. continuous

- H_φ, GNS Hilbert space.

- $S\colon H_\varphi \to H_\varphi, \quad Sx = x^*$.

Theorem. *There is a polar decomposition* $S = J\Delta^{\frac{1}{2}}$ *(isometry×positive operator) and*

$$JMJ = M'$$

$$\Delta^{it} M \Delta^{-it} = M.$$

Recall: This is a difficult theorem in analysis which serves to reduce unbounded operator theory to bounded operator theory (c.f. Weyl's form of Heisenberg's relations).

Connes' Theorem

Theorem. *Any two modular flows σ^{φ} and σ^{ψ} are related by a unitary cocyle in M:*

$$\sigma_t^{\psi}(T) = \sigma_t^{\varphi}(U_t T U_t^*).$$

In other words one has

$$\sigma\colon \mathbb{R} \times M_2(M) \to M_2(M)$$

$$\sigma_t \begin{pmatrix} a & \\ & d \end{pmatrix} = \begin{pmatrix} \sigma_t^{\varphi}(a) & \\ & \sigma_t^{\psi}(d) \end{pmatrix}$$

It follows that associated to M there is a *canonical* homomorphism

$$\mathbb{R} \longrightarrow \mathrm{Out}(M) = \mathrm{Aut}(M)/\mathrm{Inn}(M)$$

We are eventually going to distill from this the basic classification invariant for AFD factors.

KMS Condition

KMS = Kubo, Martin and Schwinger

Example:

$$\left.\begin{aligned} M &= M_n(\mathbb{C}) \\ \varphi(T) &= \mathrm{trace}(e^{-H}T) \\ H_\varphi &\cong M \end{aligned}\right\} \qquad \left\{\begin{aligned} J&\colon x \mapsto e^{-\frac{1}{2}H}x^* e^{\frac{1}{2}H} \\ \Delta&\colon x \mapsto e^{-H}xe^{H} \\ \sigma_z&\colon T \mapsto e^{izH}Te^{-izH} \end{aligned}\right.$$

Computation:

$$\begin{aligned} \varphi(\sigma_z(T_1)T_2) &= \mathrm{trace}(e^{i(z+i)H})T_1e^{-izH}T_2) \\ &= \mathrm{trace}(e^{-H}T_2e^{i(z+i)H}T_1e^{-i(z+i)H}) \\ &= \varphi(T_2\sigma_{z+i}(T_1)) \end{aligned}$$

KMS Condition:

For every T_1 and T_2 there is a continuous, bounded function F on the strip $0 \leq \mathrm{Im}(z) \leq 1$, holomorphic in the interior, with

$$\left\{\begin{aligned} F(t) &= \varphi(T_2\sigma_t(T_1)) \\ F(t+i) &= \varphi(\sigma_t(T_1)T_2) \end{aligned}\right.$$

KMS Condition, Continued

Theorem. *If σ is Tomita's modular flow associated to a normal state φ then the KMS condition is satisfied.*

Theorem. *Given a normal state φ there is a* unique *flow σ (namely Tomita's) for which the KMS condition holds.*

Example: For the Powers factors, one can check that if

$$S = \begin{pmatrix} \frac{1}{1+\lambda} & \\ & \frac{\lambda}{1+\lambda} \end{pmatrix}$$

then

$$\sigma_t(T_1 \otimes T_2 \otimes \cdots) = S^{it} T_1 S^{-it} \otimes S^{it} T_2 S^{-it} \otimes \cdots$$

satisifies the KMS condition.

The Invariant $S(M)$

G = locally compact abelian group.

Definition. The *spectrum* of an action $\alpha\colon G \times M \to M$ is the complement of the largest open set U in $\hat{G}$ such that

$$\mathrm{Supp}(\hat{f}) \subseteq U \quad \Rightarrow \quad \int_G f(g)\alpha(g)\,dg = 0$$

(here $f \in L^1(G)$).

Definition. The *Connes spectrum* of an action $\alpha\colon G \times M \to M$ is

$$S(\alpha) = \cap_p \,\mathrm{Spectrum}(\alpha\colon G \times pMp \to pMp),$$

where the $p \in M$ are nonzero α-fixed projections.

Proposition. *The Connes spectrum is a closed subgroup of $\hat{G}$, and is invariant under exterior equivalence of actions.*

Definition. The closed subgroup $S(M) \subseteq \mathbb{R}_+$ is the Connes spectrum of the modular flow (we identify $\mathbb{R}_+$ and $\hat{\mathbb{R}}$ via $\langle x, y\rangle = x^{iy}$).

A Finer Classification of Factors

- $\lambda = 0$ M is Type III_0 if $S(M) = \{1\}$.
- $0 < \lambda < 1$ M is Type III_λ if $S(M) = \lambda^{\mathbb{Z}}$.
- $\lambda = 1$ M is Type III_1 if $S(M) = \mathbb{R}_+$.

Connes' discrete decomposition:

Let $0 < \lambda < 1$. The analysis of III_λ factors is reduced to II_∞ factors and their automorphisms by a *crossed product* decomposition

$$\mathrm{III}_\lambda = \mathrm{II}_\infty \times_\alpha \mathbb{Z}$$

associated to an automorphism α of a II_∞ factor such that

$$\mathrm{trace}(\alpha(T)) = \lambda\, \mathrm{trace}(T)$$

for all $T \geq 0$. These ideas are developed to their fullest in Takesaki's theory of crossed products and duality, to be discussed next . . .

Crossed Products

$\alpha\colon G \times M \to M$, action of a locally compact abelian group on a von Neumann algebra $M \subseteq \mathcal{B}(H)$.

Definition. The *crossed product* $M \times_\alpha G \subseteq \mathcal{B}(L^2(G, H))$ is the von Neumann algebra generated by

$$\begin{cases} (Tf)(h) = \alpha_h(T) f(h), & T \in M \\ (U_g h)(h) = f(g - h), & g \in G, \end{cases}$$

or equivalently by $L^1(G, M)$ with the twisted multiplication

$$F_1 \star F_2(g) = \int_G F_1(h) \alpha_h(F_2(g - h))\, dh.$$

Definition. The *dual action* of $\hat{G}$ on $M \times_\alpha G$ is:

$$\begin{cases} \gamma(T) = T, & T \in M \\ \gamma(U_g) = \langle \gamma, g \rangle \cdot U_g, & g \in G, \end{cases}$$

Takesaki's Duality Theorem.

- $\left(M \times \alpha G\right)^{\hat{G}} \cong M$
- $M \times_\alpha G \times_{\hat{\alpha}} \hat{G} \cong M \otimes \mathcal{B}(L^2(G))$

Duality and Modular Theory

- $\sigma\colon \mathbb{R} \times M \to M$, modular group
- $N = M \times_\sigma \mathbb{R}$, crossed product
- $\hat{\sigma}\colon \mathbb{R}_+ \times N \to N$, dual action

Proposition. *The kernel of the restricted action* $\hat{\sigma}$ *on the center of* N *is precisely* $S(M)$*.*

Proposition. *The von Neumann algebra* N *has a faithful and semifinite trace* τ*, and moreover*

$$\tau(\hat{\sigma}_\lambda(T)) = \lambda\tau(T),$$

for all $T \geq 0$ *and all* $\lambda \in \mathbb{R}_+$*.*

Proof: Recall that N is generated by L^1-functions $F\colon \mathbb{R} \to M$. If F has compactly supported Fourier transform (so that F extends to an analytic function on $\mathbb{C}$) then define

$$\tau(F) = \varphi(F(i)),$$

where φ is the weight generating the modular flow σ. By the KMS condition, this is a trace.

The Flow of Weights $\mathrm{Mod}(M)$

The story so far . . .

$$M \rightsquigarrow (M, \varphi) \rightsquigarrow (M, \varphi, \sigma) \rightsquigarrow (N, \tau, \hat{\sigma})$$

From M, by choosing a faithful normal semifinite weight φ we have ultimately obtained a semifinite N, with a flow $\hat{\sigma}$ scaling the trace τ.

Crucial point: The data $(N, \tau, \hat{\sigma})$ is determined by M alone.

Lemma. *If M is a factor then the restriction of $\hat{\sigma}$ to the center of N is* ergodic *(it fixes no nontrivial element).*

Definition. The *flow of weights* or *module* for M is the restriction of $\hat{\sigma}$ to the center of N.

This is a refinement of $S(M)$, which is the kernel of this action. Notice that if the action is *transitive* then it is determined by its kernel. The intrasitive actions are (by definition) *virtual subgroups* of $\mathbb{R}_+$

Construction of Factors

Let us begin with:

N, an approximately finite dimensional II_∞ factor

$\alpha\colon \mathbb{R}_+ \times \mathrm{N} \to \mathrm{N}$, an automorphism group which scales the trace on N.

$S \subseteq \mathbb{R}_+$ a closed (virtual) subgroup

For a true subgroup S define

$$M = N^S = \{\, T \in N \,:\, \alpha_s(T) = T,\ \forall s \in S \,\}.$$

For a virtual subgroup $\mathbb{R}_+ \times A \to A$ define

$$M = N^S = \{\, T \in M \otimes A \,:\, \alpha_s(T) = T,\ \forall s \in \mathbb{R}_+ \,\}.$$

It will emerge that these are AFD factors with given module S, and that these are *all* the infinite AFD factors (excluding the trivial $M = \mathcal{B}(H)$). *Note the analogy with Galois theory.*

Injective Factors

Definition. $\mathcal{M} \subseteq \mathcal{B}(\mathcal{H})$ is *injective* if there is a completely positive splitting $\mathcal{B}(\mathcal{H}) \to \mathcal{M}$.

Fairly easy: every AFD von Neumann algebra is injective.

Stability properties: crossed products by abelian groups, increasing unions (hence AFD algebras), decreasing intersections, direct integrals.

Observe that if M is injective and $P \in M$ is a projection then PMP is injective. If M is II_∞ and P is finite then PMP is II_1.

Theorem (Connes). *There is a* unique *injective* II_1 *factor.*

Corollary. *There is a unique injective (and hence AFD)* II_∞ *factor.*

Since injective type III factors are 'reduced' to II_∞ algebras and automorphisms by modular theory, a classification comes into view . . .

Classification of Injective Factors

Theorem. *A factor is AFD if and only if it is injective.*

Theorem. *Type* III *factors which are injective/AFD are classified by their modules* $\mathrm{Mod}(\mathcal{M})$.

This is largely due to Connes, but with essential contributions from Krieger (type III_0) and Haagerup (type III_1). Of course, the modular theory of Tomita and Takesaki is also crucial.

The key methods, beyond those already sketched, build on ideas in ergodic theory to classify the automorphisms of type II algebras.