

Elliptic operators, cyclic theory, and zeta functions

John Roe
Penn State University
roe@math.psu.edu

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Abstract

In this lecture we will show how abstract elliptic operators give rise to cyclic cocycles which have strong integrality properties.

A quick review

- Noncommutative spaces — group and groupoid algebras.
- Noncommutative measure theory — von Neumann algebras — dynamic.
- Noncommutative topology — K-theory — spectral projections, indices.
- Differential topology — characteristic class (Chern-Weil) theory leads to the formalism of cyclic cohomology.

Key examples — $C(X)$, $\mathbb{C}\Gamma$, the *irrational rotation algebra*

$$A_\alpha = \langle U, V \mid UV = e^{2\pi i\alpha} VU \rangle.$$

Cyclic cohomology

For an algebra A one defines an n -cycle over A to be a package

$$A \longrightarrow \Omega^0 \xrightarrow{d} \cdots \xrightarrow{d} \Omega^n \xrightarrow{f} \mathbb{C}$$

where (Ω, d) is a differential graded algebra (not necessarily commutative) and $f: \Omega^n \rightarrow \mathbb{C}$ is a closed graded trace.

The essential information is contained in the multilinear functional $\tau(a_0, \dots, a_n) = \int a_0 da_1 \cdots da_n$, which is a *cyclic cocycle*, that is

$$\tau \circ \lambda = (-1)^\lambda \tau, \quad b\tau = 0,$$

where λ can be any cyclic permutation of a_0, \dots, a_n , and b is the ‘Hochschild coboundary’.

It turns out that b preserves the cyclic symmetry condition (Surprise!) and therefore n -cycles are in fact the (co)cycles for *cyclic cohomology theory* $HC^*(A)$.

Structure on cyclic cohomology

$HC^*(A)$ is a module over $HC^*(\mathbb{C})$, which is a polynomial ring with one 2-dimensional generator. The corresponding operation on $HC^*(A)$ is denoted $S: HC^n(A) \rightarrow HC^{n+2}(A)$.

One has the *Connes exact sequence*

$$HC^{n-1}(A) \xrightarrow{S} HC^{n+1}(A) \xrightarrow{I} HH^{n+1}(A) \xrightarrow{B} HC^n(A) \dots$$

where $HH^*(A)$ is the Hochschild cohomology of A with coefficients in A^* .

There are pairings of $HC^n(A)$ with $K_{[n]}(A)$, $[n]$ being n modulo 2, and these pairings are compatible with the S operation. The pairings are given by ‘integration of the Chern character’.

Elliptic operators

Let D be a first order elliptic operator on a compact n -dimensional manifold M . Example: $M = \mathbb{T}^n$ and D is the operator

$$\gamma_1 \frac{\partial}{\partial x^1} + \cdots + \gamma_n \frac{\partial}{\partial x^n}$$

where the Pauli matrices $\gamma_1, \dots, \gamma_n$ are generators of a Clifford algebra.

Key properties:

1. D is an unbounded selfadjoint operator on L^2 .
2. D has discrete eigenvalues tending to infinity, with finite-dimensional eigenspaces; which is to say the *resolvent operators* $(D \pm i)^{-1}$ are compact.
3. $[D, f]$ is *bounded* for a dense set of functions f on M (for instance the smooth functions).

Note. $\mathbb{Z}/2$ -grading in the even-dimensional case.

Index theory and K -homology

Properties 1 and 2 imply that $\ker D$ is a finite-dimensional space. When n is even, $\ker D$ is graded and the *index* $\text{Index } D = \dim \ker D^+ - \dim \ker D^-$ is an important invariant.

Property 3 allows us to ‘couple’ the index to a vector bundle V — there are many choices for such a coupled operator D_V but (3) implies that they all differ by bounded operators and hence have the same index. The assignment $V \mapsto \text{Index } D_V$ gives a homomorphism $K^0(M) \rightarrow \mathbb{Z}$.

Atiyah, Brown-Douglas-Fillmore, and Kasparov developed this idea into *analytic K -homology* — ‘abstract’ elliptic operators (satisfying appropriate versions of 1–3 above) are the cycles for this theory which pairs integrally with K -theory.

The definition of K -homology works even for noncommutative algebras.

Eigenvalues and zeta functions

Let μ_1, μ_2, \dots be the eigenvalues of $|D|$ arranged in increasing order. Property (2) of our list says that $\mu_j \rightarrow \infty$ as $j \rightarrow \infty$. But one can obtain a more precise quantitative form of this statement.

Theorem 1. (Weyl) *One has $\mu_j \asymp j^{1/n}$, where we recall that n is the dimension of M .*

It follows that the series defining the *zeta function*

$$\zeta_D(s) = \sum_{j=1}^{\infty} \lambda_j^{-s} = \text{Tr } |D|^{-s}$$

is convergent for $\Re s > n$. (One must make a special case of the zero eigenvalues, if any.) Note that this is the Riemann zeta function when $D = id/dx$ on the circle S^1 .

Theorem 2. *The function $\zeta_D(s)$ extends to a meromorphic function on \mathbb{C} with poles at $n, n-1, \dots$*

The proofs involve studying the heat equation.

Normalized elliptic operators

Let D be as above. To simplify assume that D is invertible, and let $F = D|D|^{-1}$ be the *phase* of D . It is a bounded operator and the results above translate to:

1. F is selfadjoint and $F^2 = 1$,
2. For all continuous functions f on M , the commutator $[F, f]$ is a *compact* operator on $H = L^2$. In fact for a dense set of functions f on M (for instance the smooth functions) this commutator belongs to the *Schatten ideal* $\mathfrak{L}^p(H)$, for every $p > n$.

Discussion. The Schatten ideal \mathfrak{L}^p is generated by those positive operators whose p 'th powers are of trace class — compare the Lebesgue space L^p . Most familiar inequalities between Lebesgue spaces have noncommutative counterparts — see B. Simon, *Trace ideals and their applications*.

Summable Fredholm modules and cycles

A (normalized) *Fredholm module* over a C^* -algebra A is given by a representation ρ of A on a Hilbert space H , together with a selfadjoint operator $F \in \mathfrak{B}(H)$ with $F^2 = 1$ and $[F, \rho(a)] \in \mathfrak{K}(H)$ for all $a \in A$.

(There is a graded variant of the above definition: H is a graded Hilbert space, the representation of A is by even operators, and F itself is odd.)

The Fredholm module (ρ, H, F) is *p -summable* if there is a dense subalgebra $\mathcal{A} \subseteq A$ such that $[F, \rho(a)]$ belongs to the Schatten ideal \mathfrak{L}^p for all $a \in \mathcal{A}$.

Theorem 3. (Connes) *Let (ρ, H, F) be a p -summable Fredholm module. Then the equalities*

$$d\alpha = [F, \alpha], \quad \int \alpha = \text{Tr}(\epsilon\alpha)$$

define a p -cycle over \mathcal{A} , called the character of the Fredholm module. (ϵ is the grading operator.)

One should take $p \equiv 0, 1 \pmod{2}$ according to grading.

Small technical improvement

Observe that the mapping

$$\mathrm{Tr}'(\alpha) = \frac{1}{2} \mathrm{Tr}(\epsilon F[F, \alpha])$$

agrees with $\mathrm{Tr}(\alpha)$ whenever $\alpha \in \mathfrak{L}^1$, but is defined for all α such that $[F, \alpha] = d\alpha$ is in \mathfrak{L}^1 . This allows us to relax slightly the order of summability required in the definition of the character. For instance, a 1-summable Fredholm module has a character in $HC^0(\mathcal{A})$ defined by

$$a \mapsto \int' a = \frac{1}{2} \mathrm{Tr}(\epsilon F[F, \rho(a)]).$$

Exercise. Check the trace property.

Pairing theorem

Let (ρ, H, F) be a Fredholm module over A and let $e \in A$ be a smooth projection. Then $\rho(e)F\rho(e)$ is a graded Fredholm operator on the space $\rho(e)H$ and we can form its index. This gives a map

$$K_0(A) \rightarrow \mathbb{Z}, \quad e \mapsto \text{Index } \rho(e)F\rho(e)$$

corresponding to $V \mapsto \text{Index } D_V$ in the commutative case.

Now suppose that F is p -summable. Then we may pair the cycle corresponding to F with the projection e according to the recipe of lecture 4, that is $\int edede \cdots de$, and we have

Theorem 4. $\text{Index}(eFe) = c_k \int e \underbrace{dede \cdots de}_{k, \text{ even}}$.

Proof If P is Fredholm, Q a parametrix modulo \mathcal{L}^p , then

$$\text{Index}(P) = \text{Tr}(1 - QP)^p - \text{Tr}(1 - PQ)^p.$$

Stability theorem

There is an ambiguity in the dimension p of the character — any integer, of appropriate parity, and greater than the degree of summability of the module, will do.

Theorem 5. *Let (ρ, H, F) be a p -summable Fredholm module and let τ_p, τ_{p+2} be its p -dimensional and $(p + 2)$ -dimensional characters. Then $[\tau_{p+2}] = c_p S[\tau_p] \in HC^{p+2}(\mathcal{A})$, where c_p is a universal constant.*

Exercise. Do the computation by hand for $p = 0$.

Thus the character is unambiguously defined in the periodic theory $HCP^*(\mathcal{A})$.

Remark. Cycles coming from summable Fredholm modules automatically satisfy the summability conditions to be higher traces — that is, they give linear functionals on the K -theory of the full C^* -algebra A .

Example I: ordinary manifolds

Let F be the Fredholm module defined by the Dirac operator D over a spin manifold M^n . It is p -summable for all $p > n$ (as we saw above), so it has a character in $HC^n(C^\infty(M))$.

Recall from lecture 4 that

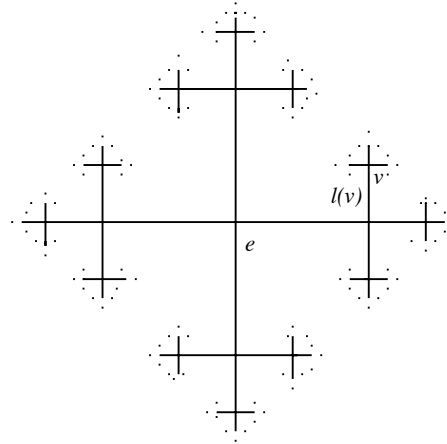
$$HC^n(C^\infty(M)) = Z_n(M) \oplus H_{n-2}(M) \oplus H_{n-4}(M) \oplus \cdots.$$

Theorem 6. *The character of the Dirac module has top-dimensional component equal to the current $(f_0, \dots, f_n) = \int_M f_0 df_1 \cdots df_n$, and the homology components are those of the Poincaré dual of the \hat{A} -genus.*

The proof uses the heat equation.

Example II: the free group

Let $\Gamma = F_2$, $A = C_r^*(\Gamma)$, $\mathcal{A} = \mathbb{C}\Gamma$, and define a Fredholm module as follows:



Graded Hilbert space $H = L^2(V) \oplus L^2(E)$.

$$F = \begin{pmatrix} 0 & U^* \\ U & 0 \end{pmatrix},$$

where U sends vertex v to edge $\ell(v)$. (1-dimensional correction gets $F^2 = 1$ on the nose.)

This Fredholm module is 1-summable. Similar finitely summable ‘dual Dirac’ constructions are possible on noncompact symmetric spaces G/K of rank one.

Kaplansky-Kadison for the free group

Lemma 1. *The character in $HC^0(\mathbb{C}\Gamma)$ of the Fredholm module defined above is equal to the von Neumann trace $\tau: \mathbb{C}\Gamma \rightarrow \mathbb{C}$.*

It follows that the map $\dim_\tau: K_0(A) \rightarrow \mathbb{R}$ is given by pairing with a Fredholm module, and so is *integer valued*. In particular the trace of a projection in A must be either 0 or 1. Since τ is faithful we get

Theorem 7. *The C^* -algebra $A = C_r^*(F_2)$ has no non trivial projections.*

Problem/Conjecture Is it true that for *any* torsionfree group, $S^n \tau$ is the character of a Fredholm module for sufficiently large n ? This would imply Kadison-Kaplansky conjecture by arguments as above.

Atiyah conjecture for the free group

Lemma 2. *Let U, T be operators on a Hilbert space H , and suppose that U commutes modulo finite rank operators with T . Then U also commutes modulo finite rank operators with the orthogonal projection operator onto the kernel of T .*

(Connes' terminology: the orthogonal projection is *quasicontinuous*.) Proof uses the fact that the set of operators of rank $\leq r$ is closed in the weak (pointwise) topology. No analog for e.g. trace class operators. Thus this is special to the free group.

Hence find that F gives a Fredholm module even over the algebra \mathfrak{A} obtained by adjoining kernel projections to $\mathcal{A} = \mathbb{C}\Gamma$. (\mathfrak{A} is a subalgebra of the von Neumann algebra A'' , but it contains elements not belonging to the C^* -algebra A .) Hence we get:

Theorem 8. (Linnell) *Any nonzero element of $\mathcal{A} = \mathbb{C}\Gamma$ is injective as an operator on $\ell^2(\Gamma)$.*

Example III: the irrational rotation algebra

As described in lecture 4, the algebra

$$A_\alpha = \langle U, V \mid UV = e^{2\pi i\alpha} VU \rangle$$

admits a canonical 2-trace

$$\tau_2(a_0, a_1, a_2) = \tau_0(a_0(\delta_1(a_1)\delta_2(a_2) - \delta_2(a_1)\delta_1(a_2))).$$

We observed the *integrality* of $\langle e, \tau_2 \rangle$ for the Powers-Rieffel projection e . Why is this?

Theorem 9. τ_2 is the character of a Fredholm module.

Extension of this to quasicontinuous elements related to the quantum Hall effect, see *Noncommutative Geometry* pp355–367.

Seeking a local index formula

The index formulae above all depend ultimately on the formula

$$\text{Index}(P) = \text{Tr}(1 - QP)^p - \text{Tr}(1 - PQ)^p$$

which is *not local*.

A truly *local* formula would *neglect* trace class operators (because, for example, pseudodifferential operators are trace class off the diagonal). Can such an index formula be found in the abstract case?

Connes-Moscovici showed that the answer is yes under a natural hypothesis. Recall that the p -summability hypothesis is equivalent to the convergence of the Dirichlet series for a certain zeta function in a right-hand half plane. It turns out that a fully local index formula can be achieved if we assume in addition that $\zeta(s)$ has an analytic continuation.

See lecture 9.