Lecture 4
Noncommutative Differential Topology

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Characteristic Numbers

M, smooth manifold

$P\colon M \to M_K(\mathbb{C})$, $\begin{cases}\text{projection-valued function}\\ \text{vector bundle}\end{cases}$

$V \subseteq M$, oriented closed submanifold

Definition. $c_V(P) = \int_V \mathrm{trace}(PdPdP\cdots dPdP)$.

Proposition. *Fixing V, the scalar $c_V(P)$ only depends on $[P] \in K^0(M)$.*

- $\mathrm{trace}(PdPdP\cdots dPdP)$ is a closed form.

- Given $P\colon I \times M \to M_n(\mathbb{C})$,

$$\int_{\partial I\times M} \mathrm{trace}(PdPdPdP) = \int_{I\times M} \mathrm{trace}(dPdPdP\cdots dPdP) = 0.$$

Remark: If $\dim(V)$ is odd then $c_V(P) = 0$ (in fact $\mathrm{trace}(PdPdP\cdots dP) = 0$).

Characteristic Numbers in Noncommutative Geometry

$\mathcal{A}$, noncommutative algebra

$P \in M_K(\mathcal{A})$, idempotent over $\mathcal{A}$

Example:

$$A = \mathbb{T}^2_\theta = \{\textstyle\sum_{m,n} a_{mn} U^m V^n\}$$

$$(UV = e^{2\pi i\theta} VU)$$

$P =$ Powers-Rieffel projection
$\quad = U^{-1} g(V) + f(V) + g(U)U$

Problem: Define characteristic numbers $c_V(P)$ as before.

If $c_V(P) = \int_V \mathrm{trace}(PdPdP \cdots dPdP)$ then

- What is V?
- What is $\int$?
- What is d?

Cycles

Definition. An n-*cycle* over an algebra A is

- a differential graded algebra Ω^* with an algebra map from A into Ω^0, and

- a closed, graded trace on Ω^n.

The trace properties:

$$\int \omega_1\omega_2 = (-1)^{\deg(\omega_1)\deg(\omega_2)} \int \omega_2\omega_1.$$

$$\int d\omega = 0.$$

Warning: It is not necesarily true that $d1 = 0$ nor that $1 \cdot \omega = \omega$, nor that $\omega_1\omega_2 = \pm\omega_2\omega_1$.

Proposition. *If* $X = (\Omega_X, \int_X)$ *is an* n*-cycle then the characteristic number*

$$c_X(P) = \int_X \mathrm{trace}(PdPdP \cdots dPdP)$$

depends only on $[P] \in K_0(A)$*.*

As before, if n is odd then $c_X(P)$ is zero.

Traces

$$\{0\text{-cycles over } A\} = \{\text{traces on } A\}$$

If P is a path of projections and $\frac{d}{dt}P = \dot{P}$ then

$$P^2 = P \;\Rightarrow\; P\dot{P} = \dot{P}P^{\perp} \;\Rightarrow\; P\dot{P}P = 0 = P^{\perp}\dot{P}P^{\perp},$$

and therefore

$$\tau(\dot{P}) = \tau(P\dot{P}P^{\perp} + P^{\perp}\dot{P}P) = 0.$$

Commutative case:

$\tau\colon C^{\infty}(M) \to \mathbb{C}$, τ = any distribution.

However, $\tau(P) = \mathrm{constant} \cdot \dim(P)$.

Noncommutative examples:

$$\tau\colon \mathbb{T}^2_{\theta} \to \mathbb{C}, \quad \tau\left(\sum a_{mn}U^mV^n\right) = a_{00}$$

$$\tau_{\lambda}\colon \mathbb{C}[G] \to \mathbb{C}, \quad \tau_{\lambda}\left(\sum a_g[g]\right) = a_e$$

Topological Invariance

A = Banach or C^*-algebra

$\mathcal{A} \subseteq A$ dense subalgebra of A

Not every cycle X over $\mathcal{A}$ induces a 'topologically invariant' map,

$$c_X \colon K_0(A) \to \mathbb{C}.$$

Example: Let $\mathcal{A} = \mathbb{C}[G] \subseteq C^*_r(G) = A$. Then for $\tau_0(\sum a_g[g]) = \sum a_g$, no corresponding map

$$\tau_0 \colon K_0(A) \to \mathbb{C}$$

is known, in general. (Note that τ_0 is even a 'state': $\tau_0(a^*a) \geq 0$.)

Sufficient condition for traces: Semicontinuity.

Sufficient condition for n-cycles:

$$\left| \int (a_1 db_1)(a_2 db_2) \cdots (a_n db_n) \right| \leq C_{b_1,\ldots,b_n} \|a_1\| \cdots \|a_n\|$$

(These are called n-*traces*.)

Cycles for the Irrational Torus

For $A = \mathbb{T}^2_\theta$ with $\theta = 0$ one can take

$$\Omega^0 = A$$

$$\Omega^1 = A \cdot ds \oplus A \cdot dt \qquad \begin{cases} dU = 2\pi i U ds \\ dV = 2\pi i V dt \end{cases}$$

$$\Omega^2 = A \cdot ds dt \qquad ds dt = -dt ds.$$

$$\int \sum_{m.n} a_{mn} U^m V^n \, ds dt = a_{00}$$

Lemma. *The same formulas define a 2-cycle on $\mathbb{T}^2_\theta$, for any θ.*

Computation: For the Powers-Rieffel projection,

$$\frac{1}{2\pi i} \int P dP dP = 1$$

(See Connes, *A survey of foliations and operator algebras.*)

Flows and Multiflows

$\alpha\colon \mathbb{R} \times A \to A,$ smooth action

$\delta(a) = \lim_{t\to 0} \frac{\alpha_t(a)-a}{t},$ derivation

$\tau\colon A \to \mathbb{C},$ invariant trace $\begin{cases} \tau(\alpha_t(a)) = \tau(a) \\ \tau(\delta(t)) = 0 \end{cases}$

$\Rightarrow \quad \Omega^0 = A$, $\Omega^1 = A \cdot dt$, $da = \delta(a)dt$

$$\int a\, dt = \tau(a)$$

$$\int a^0 da^1 = \tau(a^0 \delta(a^1))$$

Given commuting flows α_1 and α_2, we can form

$da = \delta_1(a)dt_1 + \delta_2(a)dt_2$

and (supposing τ is invariant for both flows)

$$\int a\, dt_1 dt_2 = \tau(a)$$

$$\int a^0 da^1 da^2 = \tau\big(a^0(\delta_1(a^1)\delta_2(a_2) - \delta_2(a_1)\delta_1(a_2))\big)$$

Curvature

Lemma. *The characteristic number*

$$c_\alpha(P) = \tau\Big(a^0(\delta_1(a^1)\delta_2(a_2) - \delta_2(a_1)\delta_1(a_2))\Big)$$

depends only on the outer equivalence class of the action of $\mathbb{R}^2$ on A.

Proof. Given $\alpha \sim \alpha'$, form the combined action

$$\alpha'' \colon \mathbb{R}^2 \to \mathrm{Aut}(M_2(A)),$$

$$\alpha''_t \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \alpha_t(a) & * \\ * & \alpha'_t(d) \end{pmatrix}$$

Construct from it a 2-cycle over $M_2(A)$. Since

$$\begin{pmatrix} P & 0 \\ 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 0 & 0 \\ 0 & P \end{pmatrix}$$

we get

$$c_\alpha(P) = c_{\alpha''}\begin{pmatrix} P & 0 \\ 0 & 0 \end{pmatrix} = c_{\alpha''}\begin{pmatrix} 0 & 0 \\ 0 & P \end{pmatrix} = c_{\alpha'}(P)$$

Example: Powers-Rieffel projection (compare Thom isomorphism theorem of Connes).

Cyclic n-Cocycles

$$\varphi(a^0, a^1, \ldots, a^n) = \int a^0 da^1 \cdots da^n$$

- cyclicity:

  $$\varphi(a^0, a^1, \ldots, a^n) = (-1)^n \varphi(a^1, \ldots, a^n, a^0)$$

- $b\varphi(a^0, \ldots, a^{n+1}) = 0$

  $$\begin{aligned} b\varphi(a^0, \ldots, a^{n+1}) = \varphi(&a^0 a^1, \ldots, a^{n+1}) \\ &- \varphi(a^0, a^1 a^2, \ldots, a^{n+1}) \\ &+ \ldots \\ &+ (-1)^{n+1} \varphi(a^{n+1} a^0, \ldots, a^n) \end{aligned}$$

Proposition. *Cyclic n-cocyles are precisely the functionals associated to n-cycles $(\Omega_{univ}, \int)$ on the universal differential graded algebra over A.*

Cyclic Cohomology

Lemma. *Let* φ *be a* cyclic $(n+1)$*-multlinear functional on* A*.*

- $b\varphi$ is cyclic too.
- $b^2\varphi = 0$.

Definition. $HC^n(A)$ = cyclic n-cocyles modulo cyclic coboundaries.

Pairings:

$$\langle x, \varphi \rangle \in \mathbb{C} \qquad \begin{cases} HC^{2n}(A) \otimes K_0(A) \to \mathbb{C} \\ HC^{2n+1}(A) \otimes K_1(A) \to \mathbb{C} \end{cases}$$

The latter is

$$\begin{aligned} \langle [U], \varphi \rangle &= (\varphi \times \mathrm{trace})(U^{-1}, U, U^{-1}, U \ldots, U^{-1}, U) \\ &= \textstyle\int_X \mathrm{trace}[U^{-1} dU]^n, \end{aligned}$$

assuming that $d1 = 0$.

Cyclic Cohomology and Manifolds (First Look)

For $V \subseteq M$ oriented we define

$$\varphi_V(a^0, a^1, \ldots, a^n) = \int_V a^0 da^1 \cdots da^n$$

(de Rham differential)

We obtain:

geometric n*-cycles*	$\to$	*closed de Rham currents*	$\to$	*cyclic* n*-cocyles*

However:

- b(manifold n-chain) =manifold $(n-1)$-cycle
- b(cyclic n-cochain) = cyclic $(n+1)$-cocycle
- Note the mismatch — there are some surprises!

Products and Suspension

Lemma. $\mathrm{HC}^n(\mathbb{C}) = \begin{cases} \mathbb{C} & n \text{ even} \\ 0 & n \text{ odd} \end{cases}$

Product on cycles:

$$(\Omega_{A_1}, \textstyle\int_{A_1}) \times (\Omega_{A_2}, \textstyle\int_{A_2}) = (\Omega_{A_1} \hat{\otimes} \Omega_{A_2}, \textstyle\int_{A_1} \hat{\otimes} \textstyle\int_{A_2})$$

This induces

$$\mathrm{HC}^{n_1}(A_1) \otimes \mathrm{H}^{n_2}(A_2) \to \mathrm{HC}^{n_1+n_2}(A_1 \otimes A_2)$$

Lemma. $\mathrm{HC}^*(\mathbb{C})$ *is a polynomial algebra with degree two generator* $\varphi(1,1,1) = 1$*.*

(Compare with $\mathrm{P}^\infty(\mathbb{C})$.)

Suspension:

$$\mathrm{HC}^*(\mathbb{C}) \otimes \mathrm{HC}^*(A) \to \mathrm{HC}^*(A)$$

$$S\colon \mathrm{HC}^*(A) \to \mathrm{HC}^{*+2}(A)$$

from generator of $\mathrm{HC}^2(\mathbb{C})$.

Proposition. $\langle x, \varphi \rangle = \langle x, S\varphi \rangle$

Bordism

Lemma. *If* $V \subseteq M$ *bounds then* $c_V(P) = 0$ *for all* P*.*

Proof. Stokes' Theorem.

Definition. An n-cycle $X = (\Omega_X, \int_X)$ *bounds* if there exists a d.g.a.–trace pair $W = (\Omega_W, \int_W)$, and a surjection $r\colon \Omega_W \to \Omega_X$, such that

$$\int_X r[\omega] = \int_W d\omega \qquad \text{"Stokes' Theorem"}$$

Lemma. *If* $X = (\Omega_X, \int_X)$ *bounds then* $c_X(P) = 0$*, for all projections* P*.*

Remark: This gives a natural context for showing that

$$[P] \mapsto \int_X \mathrm{trace}(PdPdP \cdots dPdP)$$

is well defined (depends only on $[P] \in K_0(A)$).

Algebraic Structure of Bordism and Suspension

$HH^n(A)$ = Hochschild cohomology (drop cyclicity condition on $(n+1)$-linear functionals; keep the same differential b).

Theorem. *A cycle bounds iff its cyclic cohomology class is in the image of the map*

$$B = AB_0 \colon HH^{n+1}(A) \to HC^n(A)$$

$$B_0\varphi(a^0, \ldots, a^n) = \varphi(1, a^0, \ldots, a^n) \pm \varphi(a^0, \ldots, a^n, 1)$$

$A =$ cyclic symmetrization

Theorem. *A cycle bounds iff its cyclic class is in the kernel of*

$$S \colon HC^n(A) \to HC^{n+2}(A)$$

Definition.

$HCP^j(A) =$ stable bordism class of cycles

$$= \varinjlim_S HC^{j+2k}(A)$$

Example: Bass Conjecture

$$\tau_0\colon \mathbb{C}[G] \to \mathbb{C}, \qquad \tau_0\left(\sum_g a_g[g]\right) = \sum_g a_g$$

$$\tau_\lambda\colon \mathbb{C}[G] \to \mathbb{C}, \qquad \tau_\lambda\left(\sum_g a_g[g]\right) = a_e$$

If G is torsion free, is

$$\tau_0 = \tau_\lambda\colon K_0(\mathbb{C}[G]) \to \mathbb{C}?$$

Stupid question: is $\tau_0 - \tau_\lambda$ a cyclic coboundary? No — there are no coboundaries in degree 0.

But we *can* try to solve

$$S(\tau_0 - \tau_\lambda) = b\psi$$

Example: If $G = \mathbb{Z}$ then take

$$\psi(a_m[m], a_n[n]) = \begin{cases} \frac{m-n}{m+n} a_m a_n & m+n \neq 0 \\ 0 & m+n = 0 \end{cases}$$

Exercise: What to do when $G = F_2$?

Other important examples arise in index theory ...

Homological Algebra

- From the map $I\colon HC^n(A) \to HH^n(A)$ associated to the inclusion of the cyclic complex into the Hochschild complex we get

$$\xrightarrow{I} HH^{n+1}(A) \xrightarrow{B} HC^n(A) \xrightarrow{S} HC^{n+2}(A)$$
$$\xrightarrow{I} HH^{n+2}(A) \xrightarrow{B}$$

- Consider $IB\colon HH^n(A) \to HH^{n+1}(A)$. One has $(IB)^2 = IBIB = I(BI)B = 0$.

- $HC^*(A) \sim$ cohomology of IB-complex (there is a spectral sequence ...)

- $bB = -Bb$ and $B^2 = 0$

- $HH^n(A)$ fits into the ordinary framework of homological algebra

Smooth Manifolds

Example: $A = C^\infty(M)$ (topological algebra)

- $HH^p(A) \cong \mathcal{D}_p(M) =$ de Rham currents.
- $IB =$ de Rham boundary
- $HCP^{ev/odd}(A) \cong H^{deRham}_{ev/odd}(M)$

Groups

Group n-cocycles:

$$c\colon G^{n+1} \to \mathbb{C}$$

$$c(gg_0, \dots gg_n) = c(g_0, \dots, g_n)$$

$$bc(g_0, \dots, g_{n+1}) = \sum (-1)^j c(g_0, \dots, \widehat{g_j}, \dots, g_{n+1}) \\ = 0$$

Cyclic cocycles from group n-cocycles:

$$\varphi_c(g_0, \dots, g_n) = \begin{cases} c(1, g_1, g_1 g_2, \dots) & \text{if } g_0 \cdots g_n = 1 \\ 0 & \text{if } g_0 \cdots g_n \neq 1 \end{cases}$$

Example:

- $f\colon G \to \mathbb{Z}$, group homomorphism
- $c(g_0, g_1) = f(g_1) - f(g_0)$, group cocycle
- $\varphi_c(g_0, g_1) = \begin{cases} f(g_1) & \text{if } g_0 g_1 = 1 \\ 0 & \text{if } g_0 g_1 \neq 1 \end{cases}$

Group Cocycles and Novikov

Example, continued:

- $\varphi_c(x, y) = \frac{1}{2\pi i}\tau_\lambda(x\delta_f(y))$, where δ_f generates the automorphism group $\alpha_t[g] = \exp(2\pi i t f(g))[g]$.

- $\varphi_c(x, y)$ is a 1-trace (it pairs with $K_1(C_r^*(G))$)

- from $c = c_1 \times \cdots \times c_j$, a product of group 1-cocycles, we get a j-trace φ_c and a map φ_{c*} on $K_j(C_r^*(G))$.

- Index formula of Connes and Moscovici: for $\alpha\colon M^n \to BG$ and for D an elliptic operator on M,

  $$\varphi_{c*}(\mu_r[D]) = (-1)^n \operatorname{constant}_j \langle ch(\sigma_D) \cdot \mathcal{T}_M \cdot \alpha^*(c), [T^*M] \rangle$$

- $\Rightarrow$ the Novikov conjecture for products of one dimensional cohomology classes