

# Non-Commutative Topology

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## Abstract

$C^*$ -algebras are the noncommutative counterpart of topological spaces. Their algebraic topology can be studied by means of K-theory.

# The big picture

Theory	Type of function	Noncommutative version
Measure Theory	Borel	Von Neumann algebra
Topology	Continuous	$C^*$ -algebra
Differential Topology	$C^\infty$	Holomorphically closed sub-algebra of a $C^*$ -algebra
Algebraic Geometry	Rational	$\mathbb{C}G$

**Today's Reading Assignment:** A. Connes, *An analogue of the Thom isomorphism for crossed products. . .*, Advances in Math **39**(1981), 31–55.

## $C^*$ -algebras

**Definition 1.** A  $C^*$ -algebra is a Banach  $*$ -algebra which is isomorphic to a subalgebra of  $\mathfrak{B}(H)$ , for some Hilbert space  $H$ .

Thus a  $C^*$ -algebra is a *norm-closed* subalgebra of  $\mathfrak{B}(H)$ ; contrast the ‘pointwise’ topology considered in lecture 2. (Since the pointwise topology is weaker than the norm topology, every von Neumann algebra is in particular a  $C^*$ -algebra; but it is not usually helpful to think like this.)

**Example** Let  $X$  be a compact metrizable space. Let  $\mu$  be a suitable measure on  $X$ . The representation  $\rho: C(X) \rightarrow \mathfrak{B}(L^2(X, \mu))$  by multiplication operators shows that  $C(X)$  is a  $C^*$ -algebra.

Note that in this example we can recover the measure by

$$\int f d\mu = \langle \rho(f)\xi, \xi \rangle$$

where  $\xi \in L^2(X, \mu)$  is the constant function 1. This points the way to an abstract characterization of  $C^*$ -algebras.

## More about $C^*$ -algebras

**Definition 2.** A  $C^*$ -algebra is a Banach  $*$ -algebra  $A$  in which the norm and involution are related by the  $C^*$ -identity  $\|a^*a\| = \|a\|^2$ .

Theory of positive linear functionals (the GNS construction) connects this abstract definition with the concrete one; Hahn-Banach arguments produce sufficiently many states to give a concrete representation of every abstract  $C^*$ -algebra. The abstract definition is useful e.g. to show that  $A/I$  is a  $C^*$ -algebra when  $A$  is a  $C^*$ -algebra and  $I$  is a closed ideal.

Commutative  $C^*$ -algebras are precisely of the form  $C_0(X)$  (Gelfand-Naimark). Consequently: If  $a \in A$  is selfadjoint, and  $f$  is continuous on the spectrum of  $a$ , then  $f(a) \in A$  also — the (continuous) *functional calculus*. If  $f$  is only (bounded) Borel then  $f(a) \in A''$  but usually  $f(a) \notin A$ .

## Group (and Groupoid) $C^*$ -algebras

Let  $\Gamma$  be a discrete group. The group ring  $\mathbb{C}\Gamma$  is a complex involutive algebra. We may obtain  $C^*$ -algebras by completing  $\mathbb{C}\Gamma$  in various norms.

- Define  $\tau: \mathbb{C}\Gamma \rightarrow \mathbb{C}$  by  $x = \sum c_g g \mapsto c_1$ . Then  $\tau$  is a positive linear functional (in fact a trace) and the associated GNS representation is the *regular representation* of  $\mathbb{C}\Gamma$  on  $\ell^2\Gamma$ . Completing in this norm gives the *reduced  $C^*$ -algebra*,  $C_{red}^*(\Gamma)$ . Notice that by construction,  $\tau$  extends to a trace on  $C_{red}^*(\Gamma)$ .
- Alternatively, for every state  $\sigma$ , let  $\|x\|_\sigma$  be the norm in the GNS representation associated to  $\sigma$ . The maximal norm on  $\mathbb{C}\Gamma$  is  $\|x\|_{\max} = \sup_\sigma \|x\|_\sigma$ , and completing in this norm gives the *maximal  $C^*$ -algebra*  $C_{max}^*(\Gamma)$ . Note that the trivial representation gives a homomorphism  $C_{max}^*(\Gamma) \rightarrow \mathbb{C}$ .

**Theorem 1.** (Hulanicki)  $C_{max}^*(\Gamma) = C_{red}^*(\Gamma)$  if and only if  $\Gamma$  is amenable.

# Projections in $C^*$ -algebras

$C^*$ -algebras are usually not stuffed full of projections.

**Kaplansky Conjecture** Let  $\Gamma$  be a torsionfree discrete group. Then  $C_{red}^*(\Gamma)$  has *no* nontrivial projections. Consequently, every selfadjoint operator in  $C_{red}^*(\Gamma)$  has connected spectrum.

Note that, in physical terms, to study the spectrum of a selfadjoint operator is to study the range of values that an observable can assume.

**Atiyah Conjecture** (simplified version) Let  $\Gamma$  be as above, let  $x \in \mathbb{C}\Gamma$ . Then the operator  $\lambda(x) \in \mathfrak{B}(\ell^2\Gamma)$  is either zero or injective. (This is really a conjecture about the  $L^2$  Betti numbers.)

The example of  $\mathbb{Z}$  shows that the Kaplansky conjecture belongs to (noncommutative) topology, whereas the Atiyah conjecture belongs to (noncommutative) (semi?)-algebraic geometry.

## Projections + Matrices = K-Theory

**Definition 3.** Let  $A$  be a  $C^*$ -algebra with unit.  $K_0(A)$  is the abelian group with one generator for each isomorphism class of finitely generated projective  $A$ -modules, and with the relations

$$[M] + [N] = [M \oplus N].$$

**Lemma 1.** If  $e, f$  are projections in a unital  $C^*$ -algebra  $A$  and  $\|e - f\| < 1$ , then there is a unitary  $u \in A$  with  $f = u^*eu$ .

**Proof** The operator  $v = ef + (1 - e)(1 - f)$  has  $ev = vf$  and satisfies  $\|1 - v\| < 1$ , so it is invertible. Use polar decomposition to replace  $v$  by its unitary part  $u$ .  $\square$

Thus one can translate the definition of  $K$ -theory as follows: the generators are homotopy classes of projections  $e = e^2 = e^*$  in matrix algebras  $M_n(A)$ . This makes the link with topology. Note that  $K$ -theory is Morita invariant.

## Where do elements of $K_0(A)$ come from?

- Spectral projections.
- *Indices* of elliptic operators ‘over  $A$ ’ (for instance, leafwise elliptic operators on a foliation).

**Remark** Even though the kernel and cokernel projections of an elliptic operator  $T$  over  $A$  belong to the von Neumann algebra  $A''$ , and not necessarily to  $A$  itself, their formal difference does define a  $K$ -theory class for  $A$ . This is another consequence of the ideas underlying the ‘heat equation’ method.

For example, the integrality of the  $L^2$  Betti numbers individually (Atiyah conjecture) is a much deeper statement than the integrality of the  $L^2$  Euler characteristic (Atiyah  $L^2$ -index theorem).

# Traces and K-theory

A *trace*  $\tau$  on a  $C^*$ -algebra  $A$  is a state that is invariant under inner automorphisms:  $\tau: A \rightarrow \mathbb{C}$  satisfies  $\tau(u^*au) = \tau(a)$  for unitary  $u$ . It will then extend to a trace on  $M_n(A)$  by the formula  $[a_{ij}] \mapsto \sum_i \tau(a_{ii})$ .

A trace  $\tau$  gives rise to a *dimension function*  $\tau_*: K_0(A) \rightarrow \mathbb{R}$ .

**Remark** In practice we often need to consider ‘unbounded traces’, e.g. on  $\mathfrak{K}(H)$ . Then delicate questions of analysis intervene.

## Smooth subalgebras

**Definition** Let  $A$  be a  $C^*$ -algebra,  $\mathcal{A} \subseteq A$  a dense subalgebra. We say that  $\mathcal{A}$  is *smooth* in  $A$  if the following is true: for every  $a \in \mathcal{A}$ , and every function  $f$  holomorphic on a neighborhood of the spectrum (in  $A$ ) of  $a$ , the element  $f(a) \in A$  in fact belongs to  $\mathcal{A}$ .

**Examples** (i)  $C^\infty(M)$ ,  $M$  a compact smooth manifold, is a smooth subalgebra of  $C(M)$ ; (ii) The subalgebra of  $A_\alpha$  consisting of those formal sums  $\sum a_{mn} U^m V^n$ , with  $\{a_{mn}\}$  of rapid decay, is smooth; (iii) Let  $\tau$  be a (densely defined) unbounded trace on a  $C^*$ -algebra  $A$ , which is *semicontinuous*; then the domain of  $\tau$  is a smooth subalgebra. For the standard trace on  $\mathfrak{K}(H)$  one obtains in this way the subalgebra  $\mathfrak{L}^1(H)$  of *trace class operators*.

(A trace  $\tau$  on  $A$  is semicontinuous if for every  $\alpha \in \mathbb{R}$ , the set  $\{a \in A : a \geq 0, \tau(a) \leq \alpha\}$  is closed in  $A$ ; compare Fatou's Lemma.)

**Theorem 2.** *Let  $\mathcal{A}$  be a smooth subalgebra of  $A$ ; then the inclusion  $\mathcal{A} \rightarrow A$  induces an isomorphism on  $K_0$ .*

# Spectral topology

Using  $K$ -theory to investigate the spectrum of a selfadjoint operator  $T$ :

1. Embed  $T$  in a  $C^*$ -algebra  $A$  with a *faithful* trace  $\tau$ , normalized so that  $\tau(1) = 1$ .
2. Compute  $K_0(A)$  and the range of  $\tau: K_0(A) \rightarrow \mathbb{R}$ .
3. Each connected component of  $\text{spectrum}(T)$  gives rise to a projection in  $A$ . Use the information about the traces of such projections.

**Elementary example** A selfadjoint  $n \times n$  matrix has no more than  $n$  eigenvalues.

## $K_1$ and Bott periodicity

The group  $K_0(A)$  has an ‘odd’ counterpart  $K_1(A)$ , generated by homotopy classes of unitary elements in matrix algebras  $M_n(A)$ . There is an elementary isomorphism  $K_1(A) \cong K_0(SA)$ , where the suspension  $SA$  is defined to be  $C_0(\mathbb{R}) \otimes A$ .

**Bott Periodicity Theorem** There is also an isomorphism  $K_0(A) \cong K_1(SA)$ .

It follows that  $K_0$  and  $K_1$  together comprise a generalized homology theory for  $C^*$ -algebras. For instance a short exact sequence

$$0 \rightarrow J \rightarrow A \rightarrow A/J \rightarrow 0$$

of  $C^*$ -algebras gives rise to a cyclic six-term exact sequence in  $K$ -theory.

**Remark** For a space  $X$  we have  $K_i(C_0(X)) = K^{-i}(X)$ , so that this is a true generalization of Atiyah-Hirzebruch  $K$ -theory.

## K-Theory for $A_\alpha$

We ask: What are the  $K$ -theory groups for the irrational rotation algebra  $A_\alpha$ ?

**Answer:** (Pimsner-Voiculescu)  $K_0(A_\alpha) = \mathbb{Z} \oplus \mathbb{Z}$ ,  $K_1(A_\alpha) = \mathbb{Z} \oplus \mathbb{Z}$ . Moreover, the dimension function given by the canonical trace on  $A_\alpha$  maps  $K_0(A_\alpha)$  to the lattice  $\mathbb{Z} \oplus \alpha\mathbb{Z} \subseteq \mathbb{R}$ .

The Pimsner-Voiculescu calculation proceeds via an exact sequence for the  $K$ -theory of a crossed product by  $\mathbb{Z}$ . More geometrically, work with the (Morita equivalent) foliation algebra for the Kronecker foliation on  $\mathbb{T}^2$ . This is a transformation groupoid algebra for a flow ( $\mathbb{R}$ -action). Thus it suffices to compute  $K$ -theory for the crossed product  $C^*$ -algebra  $A \rtimes \mathbb{R}$ .

**Theorem 3.** (Connes) *The leafwise index of  $id/dx$  gives a ‘Thom isomorphism’  $K_i(A) \rightarrow K_{i\pm 1}(A \rtimes \mathbb{R})$ .*

Note that if the  $\mathbb{R}$ -action is trivial this is just the Bott periodicity theorem.

# The Thom isomorphism

Idea of proof — reduce to the case of a trivial  $\mathbb{R}$ -action.

- If  $A = \mathbb{C}$  we always have trivial action.
- Every  $K_0$ -class for  $A$  arises from a homomorphism  $\mathbb{C} \rightarrow A$  (up to finite matrices).
- Can change the  $\mathbb{R}$ -action to an ‘exterior equivalent’ one which preserves a given smooth projection  $e$ .

**Discussion** Let the derivation  $\delta$  generate the  $\mathbb{R}$ -action. The extent to which the action fails to preserve  $e$  is measured by the *bounded* operator  $[e, \delta(e)]$ . Use this to make a ‘small’ perturbation of the *unbounded* infinitesimal generator of the  $\mathbb{R}$ -action, so as to make it commute with  $e$ .

Cannot do this for  $\mathbb{R}^2$ -actions — curvature intervenes.

## The range of the trace

Suppose given a flow on a compact space  $X$ , and let  $\mu$  be an invariant measure. One obtains from  $\mu$  a trace  $\tau$  on  $A = C(X) \rtimes \mathbb{R}$ .

**Theorem 4.** *The range of  $\dim_\tau: K_0(A) \rightarrow \mathbb{R}$  is equal to the range of the map  $H^1(X; \mathbb{Z}) \rightarrow \mathbb{R}$  given by first pairing with the vector field which generates the flow, and then integrating with respect to  $\mu$ . (exercise — this is well defined on cohomology.)*

Proof proceeds via the formula

$$\dim_\tau(\phi(u)) = \frac{1}{2\pi i} \int u^{-1} \delta(u) d\mu$$

for  $u$  unitary over  $C(X)$ . ( $\phi$  is the Thom isomorphism.)

**Corollaries:** (i) Computation of the range of the trace on  $A_\alpha$ , above; (ii) A simple  $C^*$  algebra with no nonzero idempotent — take  $X = S^3$  and a minimal diffeomorphism.

# The tangent groupoid

**Reference:** *Noncommutative Geometry* pp102–111

Let  $M$  be a compact smooth manifold. Then  $M \times M$  is (in a trivial way) a smooth groupoid. Its associated algebra  $\mathbb{C}[M \times M]$  is the algebra of smoothing operators on  $M$ . The tangent bundle  $TM$  is a smooth groupoid too: it is a bundle over  $M$  whose fibers are groups. Let us glue these together.

**Definition 4.** *The tangent groupoid to  $M$ ,  $G(M)$ , has objects  $M \times [0, 1]$ , and it is equal to  $TM$  over  $M \times \{0\}$  and to  $M \times M$  over  $M \times \{t\}$  for  $t \neq 0$ .*

This gives a smooth groupoid — local coordinates near  $(x, 0)$  may be given by

$$(x, v, t) \mapsto \begin{cases} (x, \exp_x(tv), t) & (t > 0) \\ (x, v) & (t = 0) \end{cases}$$

Notice that the decomposition of the objects  $M \times [0, 1] = M \times \{0\} \cup M \times (0, 1]$  allows us to write  $G(M) = G_0(M) \cup G_+(M)$ , the disjoint union of a closed and an open subgroupoid.

# The tangent deformation

A decomposition like this gives an exact sequence

$$0 \rightarrow C^*(G_+) \rightarrow C^*(G) \rightarrow C^*(G_0) \rightarrow 0.$$

Here  $C^*(G_+)$  has trivial  $K$ -theory and so  $K_*(C^*(G)) \cong K_*(C^*(G_0)) \cong K^*(T^*M)$ . Evaluation at some finite value of  $t$  (say  $t = 1$ ) gives a homomorphism  $C^*(G) \rightarrow \mathfrak{K}$ , and so on the level of  $K$ -theory we obtain a homomorphism

$$K^0(T^*M) \rightarrow \mathbb{Z}.$$

**Theorem 5.** *The above homomorphism is the Atiyah-Singer analytical index map.*

To prove this, construct a linear section of  $C^*(G) \rightarrow C^*(G_0)$ , using the symbol calculus for pseudodifferential operators.

$$k_t(x, y) = (2\pi)^{-n} \int e^{i\xi \cdot (x-y)/t} \sigma(x, \xi) d\xi.$$

# A noncommutative view of the index theorem

One can regard the index theorem as a reduction of the above exact sequence of noncommutative spaces to the level of *commutative* geometry. Suppose  $M$  embedded in Euclidean space  $\mathbb{R}^N$ . From this we obtain a free action of the groupoid  $G$  on the space  $\mathbb{R}^N$ . The transformation groupoid  $G \rtimes \mathbb{R}_N$  is then a proper equivalence relation, as are the subgroupoids  $G_0 \rtimes \mathbb{R}_N$  and  $G_+ \rtimes \mathbb{R}_N$ .

By Rieffel's Theorem (lecture 1) these groupoids are Morita equivalent to ordinary 'commutative' spaces.

The index theorem is thus reduced to a version of Bott periodicity relating the  $K$ -theory groups of the algebras associated to the groupoids  $G$  and  $G \rtimes \mathbb{R}_N$  — that is, to the Thom isomorphism theorem.

# Introduction to the Baum-Connes conjecture

It is natural to enquire whether every groupoid  $G$  has the property that its  $C^*$ -algebra  $C^*(G)$  is ‘K-theoretically equivalent’ to an ordinary commutative space. In a more precise form, this is the Baum-Connes conjecture: a certain *assembly map*

$$\mu: K_*^{top}(G) \rightarrow K_*(C_{red}^*(G))$$

is conjectured to be an isomorphism.

The assembly map is a generalization of the Atiyah-Singer index map.

The BC conjecture has been verified in a wide variety of examples. Recent constructions lead to groupoids  $G$  for which it fails: these exploit the tension between  $C_{max}^*$  and  $C_{red}^*$ , together with the fact that a decomposition like  $G = G_0 \cup G_+$  above in general gives an exact sequence of  $C^*$ -algebras only on the  $C_{max}^*$  level.

# Hike

For those who will be here on Friday June 23rd, a *hike* is being planned. Would those interested please sign below. Consult Guoliang Yu for details.