Lecture 2
Noncommutative Measure Theory

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Spectral Theory

$T =$ bounded selfadjoint operator on a Hilbert space H (thus $\langle Tu, v\rangle = \langle u, Tv\rangle$).

$$\mathrm{Spectrum}(T) = \{\, \lambda \in \mathbb{C} \,:\, T - \lambda I \quad \text{is singular} \,\} \subseteq \mathbb{R}.$$

Some rather amazing facts:

- The map
$$\sum_{i=1}^{n} a_i x^i \mapsto \sum_{i=1}^{n} a_i T^i$$
from polynomials to operators determines an isometric $*$-algebra homomorphism
$$C(\mathrm{Spectrum}(T)) \to \mathcal{B}(H).$$

- The map further determines an isometric $*$-algebra homomorphism from bounded Borel functions to operators. Pointwise convergence of functions corresponds to pointwise convegence of operators.

Measure classes

The image of the *Borel functional calculus homomorphism* is isomorphic to the algebra $L^\infty(X, \mu)$ for some measure μ on $X = \mathrm{Spectrum}(T)$ (assuming the Hilbert space H is separable).

The operator T determines the *measure class* $[\mu]$ (two measures belong to the same class if they have the same nullsets).

Theorem. *The measure class* $[\mu]$ *and a multiplicity function (discussed later) determine* T *up to unitary equivalence (that is, up to change of orthonormal basis in the Hilbert space).*

Example. Associated to $T = i\frac{d}{dx}$ is the Lebesgue measure class on $X = \mathbb{R}$. (This operator is *unbounded*, and requires somehat careful treatment).

Operator Algebras

Definition. Operator algebra $=$ von Neumann algebra $=$ W^*-algebra $=$ unital $*$-algebra of operators on H which is closed under the topology of pointwise convergence.

This definition is not optimal ... but it is simplest.

Example $L^\infty(X, \mu) \subseteq \mathcal{B}(L^2(X, \mu))$.

Example: If π is a unitary representation of a group on a Hilbert space then its *commutant*, the algebra of all intertwining operators, is a von Neumann algebra.

Double Commutant Theorem. *Every von Neumann algebra is the commutant of its commutant.*

Example, continued: Observe that a subspace of H is a subrepresentation of π if and only if the orthogonal projection onto it is an operator in the commutant of π.

Projections in $M \leftrightarrow$ *Subrepresentations of* π

Decomposition of Representations

Algebraic Background:

$G =$ finite group
$K =$ field
$\pi =$ finite-dimensional representation over K

Theorem. *Commutant of π = Direct sum of central simple algebras.*

Decomposition of Hilbert Space:

(X, μ) = measure space (only $[\mu]$ is relevant)
$\{H_x\}_{x \in X}$ = family of Hilbert spaces
$\mathcal{S}$ = measurable family of sections

$\Rightarrow$ We can form $H = \oplus \int_X H_x \, d\mu(x)$.

Theorem. *If M is a von Neumann algebra on H (separable) then there are canonical decompositions*

$$H = \oplus \int_X H_x \, d\mu(x) \qquad M = \oplus \int_X M_x \, d\mu(x)$$

where the M_x have trivial center.

Factors

Definition. A *factor* is a von Neumann algebra with trivial center.

Finite dimensional case: $M \cong M_n(\mathbb{C})$ = (central) simple algebra.

Message: Algebraically, factors play the role of central simple algebras over $\mathbb{C}$, in infinite dimensions (more on this later).

Definition. A factor M is of *type I* if $M \cong \mathcal{B}(H)$ (algebraically) for some H.

Are there non-type I factors? ... Yes ... Let G be a group and let M be the commutant of the left-regular representation on $\ell^2(G)$.

Theorem. *The commutant of M is the commutant of the* right-*regular representation on $\ell^2(G)$.*

Lemma. $T \mapsto \langle T\delta_e, \delta_e \rangle$ *is a faithful trace on M.*

Corollary. *If G has no finite conjugacy classes then M is not of type I.*

Comparison of Projections

Projections in $M \leftrightarrow$ Subrepresentations of π

$P_1 \sim P_2 \quad \leftrightarrow$ π_1 and π_2 are unitarily equivalent

$[P_1] \leq [P_2] \quad \leftrightarrow$ π_1 is unitarily eq. to a subrepresentation of π_2

Theorem. *This is a partial order on equivalence classes. If M is a factor then this is a* linear *order.*

P_1 is *minimal* $\quad \leftrightarrow \quad$ π_1 is irreducible

P_1 is *infinite* $\quad \leftrightarrow \quad$ π_1 is equiv. to a proper subrepresentation of itself

P_1 is *finite* $\quad \leftrightarrow \quad$ π_1 is *not* equiv. to a proper subrepresentation of itself

Proposition. M *is a factor if and only if π is* isotypic*: every two subrepresentations of π have equivalent subrepresentations.*

Proposition. *A factor has a minimal projection if and only if it is algebraically isomorphic to $\mathcal{B}(H)$ for some H (i.e. iff it is of type I).*

Decomposition of Representations

Theorem. *If M is the commutant of a commutative von Neumann algebra then*

$$M = \oplus \int_X M_x \, d\mu(x),$$

where each M_x is algebraically isomorphic to the full algebra of bounded operators on some Hilbert space

If M is the commutant of a selfadjoint operator T then X above is the spectrum of T, μ is the measure discussed earlier, and $x \mapsto \sqrt{\dim(M_x)}$ is the multiplicity function which, together with $[\mu]$, serves to characterize T.

Theorem. *If the commutant of π decomposes as above then π decomposes as a direct integral of multiples of irreducible representations.*

Remark. Among discrete groups, type I implies virtually abelian.

Types I, II and III

Relative dimension of projections in factors:

For finite projections $P_0, P_1 \in M$ (with $P_1 \neq 0$) form

$$\dim(P_0)/\dim(P_1) \in [0, \infty)$$

using a 'Euclidean algorithm'.

Absolute dimension function:

$$\dim(P) = \begin{cases} 0 & \text{if } P \text{ is zero} \\ \infty & \text{if } P \text{ is infinite} \\ \dim(P)/\dim(P_{\text{ref}}) & \text{if } P \text{ is finite} \end{cases}$$

Proposition. *There is a maximum dimension and range* $(\dim)$ *is arithmetically closed within* $[0, d_{\max}]$.

Types of Factors:

Type I	$\{0, 1, 2, \ldots, n\}$ or $\{0, 1, 2, \ldots, \infty\}$
Type II	$[0, 1]$ or $[0, \infty]$
Type III	$\{0, \infty\}$

Representations of $*$-Algebras

GNS construction:

$\mathcal{A}$ = complex $*$-algebra

$\varphi\colon \mathcal{A} \to \mathbb{C}$, $\varphi(a^*a) \geq 0$

H_ϕ = GNS Hilbert space, $\langle a, b\rangle = \varphi(b^*a)$

π_φ = GNS representation, $\pi_\varphi\colon \mathcal{A} \to \mathcal{B}(H_\varphi)$
(assuming $\varphi(a^*b^*ba) \leq C_b\varphi(a^*a)$)

Example:

$\mathcal{A} = C(X)$

$\varphi(f) = \int_X f(x)\, d\mu(x)$ (Riesz Theorem)

$H_\varphi = L^2(X, \mu)$

Infinite Tensor Products

Powers factors:

$A = \bigotimes_{j=1}^{\infty} M_2(\mathbb{C})$, CAR algebra

$\varphi = \bigotimes_{j=1}^{\infty} \varphi_\lambda$, product state

$$\varphi_\lambda \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \frac{1}{1+\lambda}(\lambda a + d)$$

R_λ = double commutant of $\pi_\varphi[A]$

Examples:

$\lambda = 0 \quad \Rightarrow$ Type I factor

$\lambda = 1 \quad \Rightarrow$ Type II factor

Theorem. *The R_λ are type III factors.*

Theorem. (R. Powers) *For $\lambda \in (0, 1)$ the R_λ are pairwise nonisomorphic type III factors.*

What Type?

Does R_λ have any non-zero, finite projections?

Is there a faithful, normal, semifinite trace on R_λ?

$$\left.\begin{array}{c} T \geq 0 \in R_\lambda \\ X = \text{Spectrum}(T) \end{array}\right\} \quad \Rightarrow \quad \left\{\begin{array}{c} \mu_T(E) = \dim(\chi_E(T)) \\ \text{Trace}(T) = \int_X x \, d\mu_T(x) \end{array}\right.$$

Obstacles:

- the trace, should it exist, need not be finite on any element of $\otimes_j M_2(\mathbb{C})$
- $\otimes_j M_2(\mathbb{C})^=$ (the norm closure) is independent of λ
- algebraic structure need not pass to weak closure: for the tracial product states it happens that $[\otimes_j M_2(\mathbb{C})]'' \cong [\otimes_j M_3(\mathbb{C})]''$

Geometric Examples of Factors

Data:

(X, μ), measure space
G, discrete group
$G \times X \to X$, measurable action

Assumptions:

action preserves class of μ
action is free

Example:

$PSL(2, \mathbb{Z}) \times P^1(\mathbb{R}) \to P^1(\mathbb{R})$.

Group-measure space construction:

$\{\ell^2(Gx)\}_{x \in X}$ = measurable field of Hilbert spaces

$M = \left\{\text{measurable families } \{T_x \in \mathcal{B}(\ell^2(Gx))\}_{x \in X/G}\right\}$

Theorem. M *is a factor iff the action is ergodic.*

Type III Factors

Theorem. *Suppose that the action is ergodic.*

- $\mathcal{M}$ *is type I iff there are atoms.*

- $\mathcal{M}$ *is type II iff there are no atoms but there is an invariant measure in the class of* μ*.*

- $\mathcal{M}$ *is type III iff there are no atoms and no invariant measures in the class of* μ*.*

This applies to the factors $\mathcal{R}_\lambda$:

$X = \prod_j \mathbb{Z}/2\mathbb{Z}$

$\mu = \prod_j \mu_\lambda$

$G = \oplus_j \mathbb{Z}/2\mathbb{Z}$

Theorem. *If* $\lambda \in (0,1)$ *then there is no invariant measure in the class of* μ*.*

Dynamics

Prelude: Let G act on X, preserving the class of μ. There is an associated unitary representation on $L^2(X, \mu)$:

$$\pi(g)f = \left[\frac{dg_*\mu}{d\mu}\right]^{\frac{1}{2}} g^*f$$

In fact there is a whole 'principal series' of unitary representations:

$$\pi_t(g)f = \left[\frac{dg_*\mu}{d\mu}\right]^{it} \pi(g)f \qquad (t \in \mathbb{R})$$

Groupoid perspective: Note that

$$\oplus \int_X \ell^2(Gx)\, dx = L^2(X \times G)$$

On $L^2(X \times G)$ define

$$U_t h = \left[\frac{dg_*\mu}{d\mu}\right]^{it} h$$

Then $U_t M U_t^* = M$, where M is the von Neumann algebra associated to the action of G on X.

Tomita's Theorem

Summary: The dynamics $\sigma_t\colon M \to M$ can be obtained directly from the weight on M associated to μ by an algebraic construction.

Theorem. *Let φ be a faithful, normal, semifinite weight on M and let H_φ be the associated GNS Hilbert space. The unbounded operator $S\colon x \mapsto x^*$ on H_φ has a polar decomposition*

$$S = J\Delta^{\frac{1}{2}}, \qquad \Delta = S^*S,$$

and $JMJ = M'$ while $\Delta^{it}M\Delta^{-it} = M$.

The Powers examples: On R_λ we have

$$\sigma_t = Ad_{U_t}\colon R_\lambda \to R_\lambda$$

$$U_t = A_\lambda^{it} \otimes A_\lambda^{it} \otimes \dots$$

$$A_\lambda = \begin{pmatrix} \frac{\lambda}{1+\lambda} & 0 \\ 0 & \frac{1}{1+\lambda} \end{pmatrix}$$ (note: $\varphi_\lambda(T) = \mathrm{trace}_2(A_\lambda T)$)

Connes' Theorem

What does Tomita's theory do?

Weights $\rightarrow$ *Automorphism groups*

Compare this to:

$T = T^* \quad \rightarrow \quad U = \exp(iT)$

Heisenberg relations $\rightarrow$ *Weyl relations*

In measure theory, equivalent measures are related by Radon-Nikodym derivatives:

$$\int_X f(x)\, d\mu = \int_X f(x)h(x)\, d\nu$$

$$\varphi(T) = \psi(A^*TA)$$

Connes' R-N Theorem. *Let* M *be a von Neumann algebra. For any* φ *and* ψ *there is a unitary 'cocycle'* $\{U_t\} \subseteq M$ *such that*

$$\sigma_t^{\varphi}(T) = \sigma_t^{\psi}(U_t^* T U_t),$$

for all $T \in M$ *and* $t \in \mathbb{R}$*.*

Invariants in the Type III situation

Definition. $T(M) = \{\, t \in \mathbb{R} : \sigma_t^{\varphi} \text{ is inner} \}$.

Lemma. $T(M)$ *is independent of* φ *— it is an invariant of* M *alone.*

Recall that for R_λ,

$$\sigma_t(T_1 \otimes T_2 \otimes \dots) = A_\lambda^{it} T_1 A_\lambda^{-it} \otimes A_\lambda^{it} T_2 A_\lambda^{-it} \otimes \dots$$

where $A_\lambda = \begin{pmatrix} \frac{\lambda}{1+\lambda} & 0 \\ 0 & \frac{1}{1+\lambda} \end{pmatrix}$.

Lemma. *This is inner for a given* $t \in \mathbb{R}$ *iff* $A_\lambda^{it} = I$*. Hence* $T(R_\lambda) = \frac{2\pi}{\log \lambda}\mathbb{Z}$*.*

Hence the invariant $T(M)$ distinguishes the Powers factors from one another.