Non-Commutative Spaces

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Abstract

In this lecture we will describe some examples of noncommutative spaces in the sense of Connes.

Commutative geometry

Theorem 1. [R. Descartes] *Euclidean geometry is the study of three functions x, y, z on $\mathbb{R}^3$.*

In other words, one can recover the geometry of the space $\mathbb{R}^3$ by studying functions on it (the coordinates). For another similar example, consider the following simple theorem: a compact space X is connected iff the algebra $C(X)$ of continuous complex-valued functions on it has no non-trivial *idempotent* (an idempotent in an algebra A is an element $a = a^2 \in A$.)

These results are special cases of a general theorem: the commutative algebra $C(X)$ completely determines the compact Hausdorff space X. (The points of X are just the homomorphisms $C(X) \to \mathbb{C}$.) If X is *locally* compact we replace $C(X)$ by $C_0(X)$, the algebra of continuous complex-valued functions that vanish at infinity.

What is noncommutative geometry?

Basic idea of NCG: we want to consider 'spaces' whose 'coordinate functions' *do not commute*. Motivation comes from the observables of quantum theory.

Key features of Connes' NCG are:

- Replace points by functions,
- Matrices,
- Positivity and Hilbert space.

For instance, we might want to consider the 'quotient space' $S^1/\mathbb{Z}$, where $\mathbb{Z}$ acts by an irrational rotation R. Classically this quotient space is structureless because the $\mathbb{Z}$ action is ergodic. But the quotient space is naturally represented by a noncommutative C^*-algebra, which has a rich structure.

Noncommutative Quotient Spaces

Consider a simple example, of two points identified by an equivalence relation.

This corresponds to the matrix algebra $M_2(\mathbb{C})$.

More generally consider an *étale* equivalence relation $\mathcal{E}$ on a compact space X. For example in the case of the irrational rotation algebra, $\mathcal{E} = S^1 \times \mathbb{Z}$ and $r, s \colon \mathcal{E} \to S^1$ are given by $r(x, n) = x$, $s(x, n) = R^n x$. It now makes sense to multiply 'continuous matrices compactly supported on $\mathcal{E}$', getting a noncommutative algebra $C_c\mathcal{E}$.

The irrational rotation algebra

Keep thinking about the example above. Clearly the algebra is generated by functions on S^1 together with an operator V corresponding to the rotation.

But functions on S^1 can be expanded in Fourier series. Using U to denote the generator (i.e. the function z) we get

Example The *irrational rotation algebra* A_α is generated by two unitaries U and V subject to the relation

$$UV = e^{2\pi i\alpha} VU.$$

This is the algebra corresponding to the *transformation groupoid* (see later) $S^1 \rtimes \mathbb{Z}$, where $\mathbb{Z}$ acts on S^1 via the irrational rotation α.

Note the word 'unitary' — We have implicitly represented the algebra on an appropriate Hilbert space. This allows us to remove the asymmetry between U and V; we complete to a C^*-*algebra*. More about this later.

Quotients and Morita equivalence

Let $\mathcal{E}$ be an étale equivalence relation on the locally compact Hausdorff space X. In order that $X/\mathcal{E}$ be a 'good' (i.e. Hausdorff) space, it is sufficient that $\mathcal{E}$ be *proper* (that is, r and s are proper maps).

Theorem 2. (Rieffel) *Let $\mathcal{E}$ be a proper étale equivalence relation on X as above. Then $C^*(\mathcal{E})$ is* Morita equivalent *to the algebra $C_0(X/\mathcal{E})$.*

(Two C^*-algebras are *Morita equivalent* if they are 'the same up to finite matrices', for instance $\mathbb{C}$ and $M_2(\mathbb{C})$ are Morita equivalent. It is a theorem that A and B are Morita equivalent iff $A \otimes \mathfrak{K}$ and $B \otimes \mathfrak{K}$ are isomorphic.)

Summary: $C^*(\mathcal{E})$ is a noncommutative generalization of $C_0(X/\mathcal{E})$. But $C^*(\mathcal{E})$ can have interesting structure even when $C_0(X/\mathcal{E})$ is highly degenerate.

The Powers-Rieffel idempotent

To show the non-triviality of the irrational rotation algebra A_α we will construct an interesting idempotent. Write elements of A_α as $\sum f_n V^n$, where the f_n are functions on the circle $\mathbb{R}/2\pi\mathbb{Z}$ (regard U as the function $e^{i\theta}$).

Choose $[a, b] \subset [0, 2\pi]$ disjoint from its $2\pi\alpha$-translate. Then let f and g_-, g_+ be functions whose graphs are illustrated below, with $g_-^2 + g_+^2 = f^2 - f$, $V g_- V^* = g_+$, and put

$$e = g_- V^* + f + g_+ V.$$

This is an idempotent (exercise!).

Measuring the size of idempotents

The linear functional

$$\tau\colon \sum f_n V^n \mapsto \frac{1}{2\pi}\int f_0$$

is a *trace* on A_α, that is, $\tau(aa') = \tau(a'a)$. We use it to measure the size of idempotents.

Clearly $\tau(1) = 1$, $\tau(e) = \alpha$ for the Powers-Rieffel idempotent e.

Theorem 3. *For any idempotent $x \in A_\alpha$ (or in a matrix algebra $M_n(A_\alpha)$) we have $\tau(x) \in \mathbb{Z} + \alpha\mathbb{Z}$.*

Why? A geometric perspective comes through foliation theory.

Groupoids 101

Definition 1. *A* groupoid *is like a group, except that composition is not defined everywhere.*

More precisely: a groupoid is comprised of a set G of arrows, a set G^0 of objects, two maps $s, r\colon G \to G^0$, and an associative composition law which allows one to form $g_1 g_2 \in G$ provided that $s(g_1) = r(g_2)$. One also requires that each object should have an associated identity arrow in G^0, and that each arrow should have a two-sided inverse.

We will usually consider locally compact *topological groupoids* (obvious definition). Such a groupoid is *étale* if r and s are local homeomorphisms, and it is *smooth* if everything is C^∞ and r and s are subimmersions.

Examples of Groupoids

- A topological *space* X can be considered as a groupoid (with only the identity arrows, i.e. $G = G^0 = X$).

- An *equivalence relation* on X defines a groupoid with objects X (there is exactly one arrow between two points of X if they are equivalent, otherwise none).

- A topological *group* can be considered as a groupoid (with only one object).

- Suppose that a group Γ acts on a space X (on the right). The *transformation groupoid* $X \rtimes \Gamma$ associated to this action has objects X, and arrows pairs (x, γ) with $s(x, \gamma) = x\gamma$, $r(x, \gamma) = x$, and

$$(x, \gamma') \cdot (x\gamma', \gamma) = (x, \gamma'\gamma).$$

- A *foliation* gives rise to a smooth groupoid (in simple cases, just the groupoid of the associated equivalence relation).

Groupoids and algebras

Let G be an étale groupoid. The *groupoid algebra* $C_c(G)$ consists of all continuous, compactly supported functions $f\colon G \to \mathbb{C}$, with composition defined by

$$(f_1 * f_2)(g) = \sum_{g_1 g_2 = g} f_1(g_1) f_2(g_2).$$

We regard this as an algebra of functions on the 'noncommutative space' defined by the groupoid. Examples: If G comes from a topological space X, we get the algebra $C_c(X)$. If G comes from a discrete group Γ we get the usual group algebra.

If G is a smooth groupoid there is also a version of this construction — one needs to replace summation by integration, but this can be done canonically using smooth densities, cf. the theory of integration on Lie groups.

The Kronecker foliation of a torus

The foliation algebra for this foliation is Morita equivalent to the irrational rotation algebra.

Each transversal to the foliation gives rise to a projection in the foliation algebra. The trace of such a projection turns out to be the *transverse measure* of the transversal.

Thus, the range $\mathbb{Z}+\alpha\mathbb{Z}$ of dimensions for projective modules is revealed as the image of the integration map $H_2(\mathbb{T}^2;\mathbb{Z}) \to \mathbb{R}$ given by the transverse measure.

Levels of smoothness

We have talked about 'algebras of functions'. But what kinds of functions?

Theory	Type of function	Noncommutative version
Measure Theory	Borel	Von Neumann algebra
Topology	Continuous	C^*-algebra
Differential Topology	C^∞	Holomorphically closed subalgebra of a C^*-algebra
Algebraic Geometry	Rational	$\mathbb{C}G$

We obtain the first three by completing the last with respect to a suitable topology.

Groupoid C^*-algebras

To get $C(S^1)$ from the trigonometric polynomials, let the algebra of polynomials act on $L^2(S^1)$ by multiplication ($=$ on $\ell^2(\mathbb{Z})$ by convolution) and take the closure in the operator norm.

Similarly one obtains a C^*-algebra from $\mathbb{C}G$ by forming the closure in a suitable operator representation on Hilbert space.

Problem. What is suitable?

The regular representation of $\mathbb{Z}$ on $\ell^2(\mathbb{Z})$ enjoys the following property: if f_n is a sequence of trigonometric polynomials, and $\|f_n\| \to 0$ in the regular representation, then $\|f_n\| \to 0$ in any other representation (exercise!). Thus the regular representation is *universal* (and thus uniquely suitable).

Not all groupoids enjoy this property. This causes trouble.