The Atiyah-Singer Index Theorem

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1 Elliptic equations

The Atiyah-Singer index theorem is concerned with the existence and uniqueness of solutions to linear partial differential equations of *elliptic type*. To understand this concept, consider the two equations

$$\frac{\partial f}{\partial x} - \frac{\partial f}{\partial y} = 0 \quad \text{and} \qquad \frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y} = 0.$$

They differ only by the factor $i = \sqrt{-1}$, but nevertheless their solutions have very different properties. Any function of the form $f(x, y) = g(x+y)$ is a solution to the first equation, but in the analogous "general solution" $g(x + iy)$ of the second equation, g must be an *analytic* function of the *complex* variable $z = x + iy$, and it was already known in the nineteenth century that such functions are very special. For example, the first equation has an infinite-dimensional set of bounded solutions, but a standard result of complex analysis implies that the only bounded solutions of the second equation are the constant functions.

The differences between the solutions of the two equations can be traced to the differences between the *symbols* of the equations, which are the polynomials in real variables ξ, η obtained by substituting $i\xi$ for $\partial/\partial x$ and $i\eta$ for $\partial/\partial y$. Thus the symbols of the two equations above are

$$i\xi - i\eta \quad \text{and} \qquad i\xi + \eta$$

respectively. An equation is said to be *elliptic* if its symbol is zero only when $\xi = \eta = 0$; thus the second equation is elliptic but the first is not. The fundamental *regularity theorem*, which is proved using Fourier analysis, states that an elliptic partial differential equation (subject to suitable boundary conditions, if needed) has a finite-dimensional solution space.

2 Topology of elliptic equations and the Fredholm index

Consider now the general first-order linear partial differential equation

$$a_1 \frac{\partial f}{\partial x_1} + \cdots + a_n \frac{\partial f}{\partial x_n} + bf = 0,$$

in which f is a vector-valued function and the coefficients a_j and b are complex matrix-valued functions. It is *elliptic* if its *symbol*

$$i\xi_1 a_1(x) + \cdots + i\xi_n a_n(x)$$

is an invertible matrix for all $\xi \neq 0$ (and for every value of x). The regularity theorem applies in this generality, and it allows us to form the *Fredholm index* of an elliptic equation (with suitable boundary conditions), which is the number of linearly independent solutions of the equation minus the number of linearly independent solutions of the *adjoint equation*

$$a_1^* \frac{\partial f}{\partial x_1} + \cdots + a_n^* \frac{\partial f}{\partial x_n} - b^* f = 0.$$

The Fredholm index is a *topological invariant* of elliptic equations, which means that continuous variations in the coefficients of an elliptic equation leave the Fredholm index unchanged. As a result, it becomes very interesting to determine the topological structure of the set of all possible elliptic equations, since the Fredholm index is constant on each component of this set. This observation, that Fredholm indices provide topological information, was made by Gelfand in the 1950s. It lies at the root of the index theorem.

3 An example

To see in more detail how topology can be used to determine the Fredholm index of an elliptic equation, let us consider a specific example. Consider elliptic equations for which the coefficients $a_j(x)$ and $b(x)$ are *polynomial* functions of x, with a_j of degree $m-1$ and b of degree m. The expression

$$i\xi_1 a_1(x) + \cdots + i\xi_n a_n(x) + b(x)$$

is then a polynomial in both x and ξ. Let us strengthen the hypothesis of ellipticity by assuming that the terms in this expression that have degree m (jointly in x and ξ) define an invertible matrix whenever *either* x *or* ξ is nonzero. Let us also agree to consider only solutions f of the equation or its adjoint which are *square-integrable*, which means that

$$\int |f(x)|^2\, dx < \infty.$$

All these extra hypotheses are types of boundary conditions (the behaviors of the equation and its solutions at infinity are controlled), and collectively they imply that the Fredholm index is defined and finite.

A simple example is the equation

$$\frac{df}{dx} + xf = 0. \tag{1}$$

The general solution to this ordinary differential equation is the 1-dimensional space of multiples of the function $e^{-x^2/2}$, all of which are square-integrable. By contrast, the solutions of the adjoint equation

$$\frac{df}{dx} - xf = 0 \tag{2}$$

are multiples of $e^{+x^2/2}$, which is not square-integrable. Thus the index of this differential equation is equal to 1.

Returning to the general equation, the degree m terms in

$$i\xi_1 a_1(x) + \cdots + i\xi_n a_n(x) + b(x)$$

determine a map from the unit sphere in (x, ξ)-space to the set $GL(k, \mathbb{C})$ of invertible $k \times k$ matrices. Moreover, every such map comes from an elliptic equation (of a generalized type). It therefore becomes important to determine the topological structure of the space of all maps from the sphere S^{2n-1} into $GL(k, \mathbb{C})$.

A remarkable theorem of Bott provides the answer. The *Bott periodicity theorem* associates an integer, which we shall call the *Bott invariant*, to each map $S^{2n-1} \to GL(k, \mathbb{C})$. Furthermore, Bott's theorem asserts that provided $k \geq n$, one such map can be continuously deformed into another one if and only if their Bott invariants agree. In the special case $n = k = 1$, so that we are dealing with maps from the one-dimensional circle into the nonzero complex numbers, or in other words closed paths in $\mathbb{C}$ that do not pass through the origin, the Bott invariant is just the classical *winding number*, which measures the number of times a such a path winds around the origin. We may therefore regard the Bott invariant as a generalized winding number.

The index theorem for the equations of the type we are considering in this section asserts that the Fredholm index of an elliptic equation is equal to the Bott invariant of its symbol. For instance, in the case of the simple example (1) considered above, the symbol $x + i\xi$ corresponds to the identity map from the unit circle in (x, ξ)-space to the unit circle in $\mathbb{C}$. Its winding number is therefore equal to 1, in agreement with our computation of the index.

The proof of the index theorem depends strongly on Bott periodicity. Because elliptic equations are classified topologically by the Bott invariant, and because the Bott invariant and the Fredholm index have analogous algebraic properties, one need only verify the theorem in a single example: that corresponding to a symbol with Bott invariant 1. It turns out that this *Bott generator* can be represented by an n-dimensional generalization of our example (1), and a computation in this case completes the proof.

4 Elliptic equations on manifolds

It is possible to define elliptic equations not just for functions f of n variables, but for functions defined on a MANIFOLD. Particularly accessible to analysis are the elliptic equations on *closed* manifolds, that is, manifolds which are finite in extent and which have no boundary. For closed manifolds it is not necessary to specify any boundary conditions in order to obtain the basic regularity theorem for elliptic equations. As a result, every elliptic partial differential equation on a closed manifold has a Fredholm index.

The general Atiyah-Singer index theorem has the same form as the particular case that we studied above. One builds out of the symbol an invariant called the *topological index*, which generalizes the Bott invariant. The index theorem then asserts that the topological index constructed from the symbol of an equation is equal to the Fredholm or "analytical" index of the equation itself.

The key to the proof is a counterpart for manifolds of the theorem of Bott. This is obtained using K-THEORY, which is a branch of ALGEBRAIC TOPOLOGY invented by Atiyah and Hirzebruch. Once this is available, the proof of the general Atiyah-Singer index theorem proceeds in the same way as proof of the special index theorem of the last section. The Fredholm index and the topological index are both topological invariants of elliptic equations. By computing a small number of fundamental examples and by showing that both functions have similar algebraic properties, Atiyah and Singer proved that the two invariants are equal.

Although the proof sketched above makes use of K-theory, the formula for the topological index can be translated into other terms that do not mention K-theory explicitly. In this way one obtains the index formula

$$\text{Index} = \int_M I_M \cdot \text{ch}(\sigma).$$

The term I_M is a DIFFERENTIAL FORM determined by the curvature of the manifold M on which the equation is defined. The term $\text{ch}(\sigma)$ is a differential form obtained from the symbol of the equation.

5 Applications

In order to prove the index theorem, Atiyah and Singer were obliged to study a very broad class of generalized elliptic equations. However the applications they first had in mind were related to the simple equation with which we began this article. Solutions of the equation

$$\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y} = 0$$

are precisely the analytic functions of the complex variable $z = x + iy$. There is a counterpart to this equation on any RIEMANN SURFACE and the Atiyah-Singer index formula, applied in this instance, is equivalent to a foundational result about the geometry of surfaces called the *Riemann-Roch theorem.* The Atiyah-Singer index theorem then gives a means to generalize the Riemann-Roch theorem to a complex manifold of any dimension.

The Atiyah-Singer index theorem also has important applications outside of complex geometry. The simplest example involves the elliptic equation $d\omega + d^*\omega = 0$ on differential forms on a manifold M. The Fredholm index may be identified with the *Euler characteristic* of M—the alternating sum of the numbers of r-dimensional cells in a cell decomposition of M. For 2-dimensional manifolds the Euler characteristic is the familiar quantity $V - E + F$. In the 2-dimensional case, the index theorem reproduces the GAUSS-BONNET THEOREM, that the Euler characteristic is a multiple of the total Gaussian curvature.

Even this simple case, the index theorem can be used to produce topological restrictions on the ways a manifold can curve. Many important applications of the index theorem proceed in the same direction. For example, Hitchin used a refined index theorem to show that there is a 9-dimensional manifold homeomorphic to the sphere, but which carries *no* metric of positive curvature.

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