Now let us return to Eq. (17) and consider the case of resonance, where $\omega = \omega_0$; that is, the frequency of the forcing function is the same as the natural frequency of the system. Then the nonhomogeneous term $F_0 \cos \omega t$ is a solution of the homogeneous equation. In this case the solution of Eq. (17) is

$$u = c_1 \cos \omega_0 t + c_2 \sin \omega_0 t + \frac{F_0}{2m\omega_0} t \sin \omega_0 t. \tag{24}$$

Because of the term $t \sin \omega_0 t$, the solution (24) predicts that the motion will become unbounded as $t \to \infty$ regardless of the values of c_1 and c_2; see Figure 3.8.8 for a typical example. Of course, in reality, unbounded oscillations do not occur. As soon as u becomes large, the mathematical model on which Eq. (17) is based is no longer valid, since the assumption that the spring force depends linearly on the displacement requires that u be small. As we have seen, if damping is included in the model, the predicted motion remains bounded; however, the response to the input function $F_0 \cos \omega t$ may be quite large if the damping is small and ω is close to ω_0.

PROBLEMS

In each of Problems 1 through 4 write the given expression as a product of two trigonometric functions of different frequencies.

1. $\cos 9t - \cos 7t$
2. $\sin 7t - \sin 6t$
3. $\cos \pi t + \cos 2\pi t$
4. $\sin 3t + \sin 4t$
5. A mass weighing 4 lb stretches a spring 1.5 in. The mass is displaced 2 in. in the positive direction from its equilibrium position and released with no initial velocity. Assuming that there is no damping and that the mass is acted on by an external force of $2 \cos 3t$ lb, formulate the initial value problem describing the motion of the mass.
6. A mass of 5 kg stretches a spring 10 cm. The mass is acted on by an external force of $10 \sin(t/2)$ N (newtons) and moves in a medium that imparts a viscous force of 2 N when the speed of the mass is 4 cm/s. If the mass is set in motion from its equilibrium position with an initial velocity of 3 cm/s, formulate the initial value problem describing the motion of the mass.
7. (a) Find the solution of Problem 5.
   (b) Plot the graph of the solution.
   (c) If the given external force is replaced by a force $4 \sin \omega t$ of frequency ω, find the value of ω for which resonance occurs.
8. (a) Find the solution of the initial value problem in Problem 6.
   (b) Identify the transient and steady state parts of the solution.
   (c) Plot the graph of the steady state solution.
   (d) If the given external force is replaced by a force of $2 \cos \omega t$ of frequency ω, find the value of ω for which the amplitude of the forced response is maximum.
9. If an undamped spring–mass system with a mass that weighs 6 lb and a spring constant 1 lb/in is suddenly set in motion at $t = 0$ by an external force of $4 \cos 7t$ lb, determine the position of the mass at any time and draw a graph of the displacement versus t.
10. A mass that weighs 8 lb stretches a spring 6 in. The system is acted on by an external force of $8 \sin 8t$ lb. If the mass is pulled down 3 in and then released, determine the position of the mass at any time. Determine the first four times at which the velocity of the mass is zero.

11. A spring is stretched 6 in by a mass that weighs 8 lb. The mass is attached to a dashpot mechanism that has a damping constant of 0.25 lb·s/ft and is acted on by an external force of $4\cos 2t$ lb.

   (a) Determine the steady state response of this system.

   (b) If the given mass is replaced by a mass m, determine the value of m for which the amplitude of the steady state response is maximum.

12. A spring–mass system has a spring constant of 3 N/m. A mass of 2 kg is attached to the spring, and the motion takes place in a viscous fluid that offers a resistance numerically equal to the magnitude of the instantaneous velocity. If the system is driven by an external force of $(3\cos 3t - 2\sin 3t)$ N, determine the steady state response. Express your answer in the form $R\cos(\omega t - \delta)$.

13. In this problem we ask you to supply some of the details in the analysis of a forced damped oscillator.

   (a) Derive Eqs. (10), (11), and (12) for the steady state solution of Eq. (8).

   (b) Derive the expression in Eq. (13) for Rk/F_0.

   (c) Show that ω^2_{max} and R_{max} are given by Eqs. (14) and (15), respectively.

14. Find the velocity of the steady state response given by Eq. (10). Then show that the velocity is maximum when $\omega = \omega_0$.

15. Find the solution of the initial value problem

$$u'' + u = F(t), \qquad u(0) = 0, \quad u'(0) = 0,$$

where

$$F(t) = \begin{cases} F_0 t, & 0 \le t \le \pi, \\ F_0(2\pi - t), & \pi < t \le 2\pi, \\ 0, & 2\pi < t. \end{cases}$$

Hint: Treat each time interval separately, and match the solutions in the different intervals by requiring u and u' to be continuous functions of t.

16. A series circuit has a capacitor of 0.25×10^{-6} F, a resistor of 5×10^3 Ω, and an inductor of 1 H. The initial charge on the capacitor is zero. If a 12-volt battery is connected to the circuit and the circuit is closed at $t = 0$, determine the charge on the capacitor at $t = 0.001$ s, at $t = 0.01$ s, and at any time t. Also determine the limiting charge as $t \to \infty$.

17. Consider a vibrating system described by the initial value problem

$$u'' + \tfrac{1}{4}u' + 2u = 2\cos\omega t, \qquad u(0) = 0, \quad u'(0) = 2.$$

   (a) Determine the steady state part of the solution of this problem.

   (b) Find the amplitude A of the steady state solution in terms of ω.

   (c) Plot A versus ω.

   (d) Find the maximum value of A and the frequency ω for which it occurs.

18. Consider the forced but undamped system described by the initial value problem

$$u'' + u = 3\cos\omega t, \qquad u(0) = 0, \quad u'(0) = 0.$$

   (a) Find the solution $u(t)$ for $\omega \ne 1$.

   (b) Plot the solution $u(t)$ versus t for $\omega = 0.7$, $\omega = 0.8$, and $\omega = 0.9$. Describe how the response $u(t)$ changes as ω varies in this interval. What happens as ω takes on values closer and closer to 1? Note that the natural frequency of the unforced system is $\omega_0 = 1$.

19. Consider the vibrating system described by the initial value problem

$$u'' + u = 3\cos\omega t, \qquad u(0) = 1, \quad u'(0) = 1.$$

(a) Find the solution for $\omega \neq 1$.

(b) Plot the solution $u(t)$ versus t for $\omega = 0.7$, $\omega = 0.8$, and $\omega = 0.9$. Compare the results with those of Problem 18; that is, describe the effect of the nonzero initial conditions.

20. For the initial value problem in Problem 18 plot u' versus u for $\omega = 0.7$, $\omega = 0.8$, and $\omega = 0.9$. Such a plot is called a phase plot. Use a t interval that is long enough so that the phase plot appears as a closed curve. Mark your curve with arrows to show the direction in which it is traversed as t increases.

Problems 21 through 23 deal with the initial value problem

$$u'' + 0.125u' + 4u = F(t), \qquad u(0) = 2, \quad u'(0) = 0.$$

In each of these problems:

(a) Plot the given forcing function $F(t)$ versus t, and also plot the solution $u(t)$ versus t on the same set of axes. Use a t interval that is long enough so the initial transients are substantially eliminated. Observe the relation between the amplitude and phase of the forcing term and the amplitude and phase of the response. Note that $\omega_0 = \sqrt{k/m} = 2$.

(b) Draw the phase plot of the solution; that is, plot u' versus u.

21. $F(t) = 3\cos(t/4)$

22. $F(t) = 3\cos 2t$

23. $F(t) = 3\cos 6t$

24. A spring–mass system with a hardening spring (Problem 32 of Section 3.7) is acted on by a periodic external force. In the absence of damping, suppose that the displacement of the mass satisfies the initial value problem

$$u'' + u + \tfrac{1}{5}u^3 = \cos\omega t, \qquad u(0) = 0, \quad u'(0) = 0.$$

(a) Let $\omega = 1$ and plot a computer-generated solution of the given problem. Does the system exhibit a beat?

(b) Plot the solution for several values of ω between 1/2 and 2. Describe how the solution changes as ω increases.

25. Suppose that the system of Problem 24 is modified to include a damping term and that the resulting initial value problem is

$$u'' + \tfrac{1}{5}u' + u + \tfrac{1}{5}u^3 = \cos\omega t, \qquad u(0) = 0, \quad u'(0) = 0.$$

(a) Plot a computer-generated solution of the given problem for several values of ω between 1/2 and 2, and estimate the amplitude R of the steady response in each case.

(b) Using the data from part (a), plot the graph of R versus ω. For what frequency ω is the amplitude greatest?

(c) Compare the results of parts (a) and (b) with the corresponding results for the linear spring.

REFERENCES Coddington, E. A., *An Introduction to Ordinary Differential Equations* (Englewood Cliffs, NJ: Prentice-Hall, 1961; New York: Dover, 1989).

There are many books on mechanical vibrations and electric circuits. One that deals with both is

Close, C. M., and Frederick, D. K., *Modeling and Analysis of Dynamic Systems* (3rd ed.) (New York: Wiley, 2001).