Solution

Math 251
Section: 9

Quiz 1
Spring 2009

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All work must be shown to receive full credit!

1. A 1000m^3 room initially contains fresh air. At $t = 0$, a faulty heating system causes gas containing 10% carbon monoxide to be pumped into the room at a rate of 10m^3 per minute. A well-mixed air is vented out at the same rate.

(a) Write a differential equation, and give the initial condition, that describe this event.

Notations:

A - ammount of Air

M - ammount of Carbon Monoxide

t - time

$$A'(t) = r_{in} - r_{out} = 0$$

$$r_{in} = 10\,\frac{m^3}{min} = r_{out}$$

$$\Downarrow$$

$$A(t) = A_0 = 1000\,m^3$$

$$c_{in} = .1, \quad c_{out} = \frac{M(t)}{A(t)} = \frac{1}{1000\,m^3} M(t)$$

$$M'(t) = r_{in} c_{in} - r_{out} c_{out} = 10 \cdot .1 - 10\,\frac{1}{1000} M(t)$$

$$\boxed{\begin{cases} M'(t) = 1 - \dfrac{M(t)}{100} \\ M(0) = 0 \end{cases}}$$

(b) Solve the initial value problem.

Integrating factor:

$$M'(t) + \frac{1}{100} M(t) = 1 \quad \Big| \cdot \mu(t)$$

$$\mu(t) M'(t) + \underbrace{\frac{1}{100}\mu(t)}_{= \mu'(t)} M(t) = \mu(t)$$

$$\mu'(t) = \frac{1}{100}\mu(t) \implies \mu(t) = e^{\frac{t}{100}}$$

$$\mu(t) M'(t) + \mu(t)' M(t) = \left(\mu(t) M(t)\right)'$$

$$\left(\mu(t) M(t)\right)' = \mu(t)$$

Integrate

$$\int \left(e^{\frac{t}{100}} M(t)\right)' dt = \int e^{\frac{t}{100}}\, dt$$

$$e^{\frac{t}{100}} M(t) = 100\, e^{\frac{t}{100}} + C \quad \Big| \div e^{\frac{t}{100}}$$

$$M(t) = 100 + C e^{\frac{-t}{100}}$$

From initial conditions

$$M(0) = 0 = 100 + C \cdot 1 \implies C = -100$$

Hence, solution

$$\boxed{M(t) = 100 - 100\, e^{\frac{-t}{100}}}$$

(c) A carbon monoxide detector in the room is triggered when the carbon monoxide reaches 1%. Find the time when the detector will sound the alarm.

Find t such that the concentration

$$\frac{M(t)}{A(t)} = \frac{M(t)}{A_0} = .01 \implies \frac{100\left(1 - e^{\frac{-t}{100}}\right)}{1000} = .01 \implies$$

$$\implies 1 - e^{\frac{-t}{100}} = .1 \implies e^{\frac{-t}{100}} = .9 \implies$$

$$\implies \frac{-t}{100} = \ln(.9) \implies \boxed{t = -100 \ln(.9)}$$

or

$$t = 100 \ln\left(\frac{10}{9}\right)$$