1 Some properties of Laplace transform

One of the more important properties of the Laplace transform is how it treats the differentiation.

1.1 Derivatives

$$\begin{aligned}\mathcal{L}\{f'(t)\}(s) &= \int_0^\infty e^{-st} f'(t)\ dt = e^{-st} f'(t)\Big|_{t=0}^{\infty} + s\int_0^\infty e^{-st} f(t)\ dt = \\ &= s\mathcal{L}\{f(t)\}(s) - f(0).\end{aligned} \tag{1}$$

We can apply formula (1) n times to compute Laplace transform of the n-th derivative of a function

$$\begin{aligned}\mathcal{L}\{f^{(n)}(t)\}(s) &= \mathcal{L}\left\{\frac{d}{dt} f^{(n-1)}(t)\right\}(s) = \\ &= s\mathcal{L}\{f^{(n-1)}(t)\}(s) - f^{(n-1)}(0) = \\ &= s\Big[s\mathcal{L}\{f^{(n-2)}(t)\}(s) - f^{(n-2)}(0)\Big] - f^{(n-1)}(0) = \\ &= s\Big[s\Big[s[\ldots] - f^{(n-3)}(0)\Big] - f^{(n-2)}(0)\Big] - f^{(n-1)}(0) = \\ &= s^n \mathcal{L}\{f(t)\}(s) - s^{n-1} f(0) - \ldots - s f^{(n-2)}(0) - f^{(n-1)}(0).\end{aligned}$$

1.2 Polynomials

$$\begin{aligned}\mathcal{L}\{t^n\}(s) &= \int_0^\infty e^{-st} t^n\ dt = \frac{-1}{s}\int_0^\infty \frac{d}{dt}\left(e^{-st}\right) t^n\ dt = \\ &= \frac{-1}{s} e^{-st} t^n \Big|_{t=0}^{\infty} + \frac{n}{s}\int_0^\infty e^{-st} t^{n-1}\ dt = \frac{n}{s}\mathcal{L}\{t^{n-1}\}(s) = \ldots = \\ &= \frac{n}{s}\cdot\frac{n-1}{s}\cdot\ldots\cdot\frac{1}{s} = \frac{n!}{s^n}.\end{aligned} \tag{2}$$

1.3 Multiplication by an exponential

$$\begin{aligned}\mathcal{L}\{e^{ct} f(t)\}(s) &= \int_0^\infty e^{-st} e^{ct} f(t)\ dt = \int_0^\infty e^{-(s-c)t} f(t)\ dt = \\ &= \mathcal{L}\{f(t)\}(s-c).\end{aligned}$$

1.4 Multiplication by a polynomial

$$\begin{aligned}\mathcal{L}\{t^n f(t)\}(s) &= \int_0^\infty e^{-st} t^n f(t)\ dt = (-1)^n \int_0^\infty (-t)^n e^{-st} f(t)\ dt = \\ &= (-1)^n \int_0^\infty \frac{d^n}{ds^n}\left(e^{-st}\right) f(t)\ dt = (-1)^n \frac{d^n}{ds^n} \int_0^\infty e^{-st} f(t)\ dt = \\ &= (-1)^n \frac{d^n}{ds^n} \mathcal{L}\{f(t)\}(s).\end{aligned}$$

1.5 Multiplication by a step function

The step function $u_c(t)$ is defined as

$$u_c(t) = \begin{cases} 0 & \text{if } t < c \\ 1 & \text{if } t \geq c \end{cases} \tag{3}$$

Then

$$\begin{aligned}\mathcal{L}\{u_c(t) f(t)\}(s) &= \int_0^\infty e^{-st} u_c(t) f(t)\ dt = \int_c^\infty e^{-st} f(t)\ dt = \begin{bmatrix} \tau = t - c \\ t = \tau + c \end{bmatrix} = \\ &= \int_0^\infty e^{-s(\tau+c)} f(\tau + c)\ d\tau = e^{-sc} \int_0^\infty e^{-s\tau} f(\tau + c)\ d\tau = \\ &= e^{-sc} \mathcal{L}\{f(t+c)\}(s)\end{aligned}$$