MATH 251

Examination I

February 26, 2009

Name:________________

Student Number:________________

Section:________________

This exam has 15 questions for a total of 100 points. **In order to obtain full credit for partial credit problems, all work must be shown. Credit will not be given for an answer not supported by work.** The point value for each question is in parentheses to the right of the question number.

YOU MAY NOT USE A CALCULATOR ON THIS EXAM. PLEASE TURN OFF AND PUT AWAY YOUR CELL PHONE.

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1 :	
2 thru 11:	
12:	
13:	
14:	
15:	
Total:	

1. (9 points) Consider the list of differential equations below.

A. $y' = y^3 - y^2$

B. $y''' - y^2 = \sin 3t$

C. $y'' + e^t y' = 1$

D. $t^2 + y + (t + y^2)y' = 0$

E. $y'' - t^3 y' + e^{2t} y = 0$

F. $y'' - 2y + 5y = 9 - e^y$

G. $y' + 2y = \pi$

H. $y' - t^2 y = t \ln t$

For each part, write down the letter corresponding to the equation on the list with the specified properties. There is only one correct answer to each part.

(a) (3 points) First order linear equation that is not separable.

H

(b) (3 points) Exact equation that is not separable.

D

(c) (3 points) Second order homogeneous linear equation.

E

2. (5 points) Consider the initial value problem

$$\sin(t)\,y'' + \tan(t)\,y' + t\,y = e^t, \qquad y\left(\frac{\pi}{4}\right) = \frac{4\pi}{3}, \qquad y'\left(\frac{\pi}{4}\right) = -\frac{\pi}{4}.$$

Without solving the equation, what is the largest interval in which a unique solution is guaranteed to exist?

In the standard form

(a) $(-\infty, \infty)$

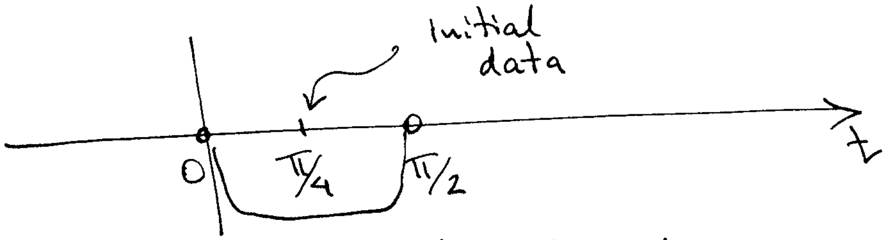
(b) $(-\frac{\pi}{2}, \frac{\pi}{2})$

(c) $(0, \frac{\pi}{2})$

(d) $(\frac{\pi}{2}, \frac{3\pi}{2})$

$$y'' + \underbrace{\frac{1}{\cos(t)}}\, y' + \underbrace{\frac{t}{\sin(t)}}\, y = \underbrace{e^t/\sin t}$$

problems at $(t = \pi/2 + \pi n)$ $(t = \pi n)$



largest interval

3. (5 points) Which pair of functions below <u>cannot</u> be a fundamental set of solutions?

(a) $4, \qquad 2 + 3t$

(b) $\cos 5t, \qquad -2\sin 5t$

(c) $e^{-t}, \qquad 4te^{-t}$

(d) $2e^{-5t}, \qquad -6e^{-5t}$ ← These are linearly dependent.

4. (5 points) Find the solution of the initial value problem

$$y' = \frac{e^{4t}}{y}, \qquad y(0) = -2.$$

Separable

(a) $y = -\sqrt{\frac{1}{2}e^{4t} + \frac{7}{2}}$

(b) $y = \frac{1}{2}e^{4t} - \frac{5}{2}$

(c) $y = \sqrt{\frac{1}{2}e^{4t} - \frac{9}{2}}$

(d) $y = -\sqrt{2e^{4t}}$

$$y\,y' = e^{4t}$$

integrate in t

$$\int y\,y'\,dt = \int e^{4t}\,dt$$

$$\frac{1}{2}y^2 = \frac{1}{4}e^{4t} + C$$

From initial conditions $y(0) = -2$

$$\frac{1}{2}(-2)^2 = \frac{1}{4}e^0 + C \Rightarrow C = \frac{4}{2} - \frac{1}{4} = \frac{7}{4}$$

$$\boxed{y = -\sqrt{\frac{1}{2}e^{4t} + \frac{7}{2}}.}$$

Note, sign $(-)$ is determined by the initial conditions.

5. (5 points) Find the general solution of the exact equation

$$\underbrace{3x^2y^3 - ye^{xy}}_{F_x(x,y)} + \underbrace{(3x^3y^2 - xe^{xy} + 2)}_{F_y(x,y)}\,y' = 0.$$

(a) $x^3y^3 - e^{xy} + 2y = C$

(b) $2x^3y^3 - e^{xy} = C$

(c) $9x^2y^2 - e^{xy} - xye^{xy} = C$

(d) $\dfrac{ye^{xy} - 3x^2y^3}{3x^3y^2 - xe^{xy} + 2} = C$

$$F(x,y) = x^3y^3 - e^{xy} + g_1(y)$$

$$F(x,y) = x^3y^3 - e^{xy} + 2y + g_2(x)$$

Combining

$$F(x,y) = x^3y^3 - e^{xy} + 2y + C$$

6. (5 points) Suppose a mass-spring system described by the equation

$$u'' + k\,u = 2\cos 4t - \sin 4t = R\cos(4t - \delta)$$

is undergoing resonance. What is the value of the spring constant k?

(a) 0

(b) 2

(c) 4

(d) 16

Resonance is observed when the frequency of the external force $\omega = 4$ matches the natural frequency of the system $\omega_0 = \sqrt{\frac{k}{1}}$

Hence equation for k:

$$\sqrt{k} = 4 \Rightarrow \underline{k = 4^2 = 16}$$

7. (5 points) A 100-liter vat initially contains 80 liters of 2 grams/liter sodium hydroxide solution. At $t = 0$, sodium hydroxide solution with a concentration of 5 grams/liter begins to flow into the vat at the rate of 2 liters/min. The thoroughly mixed content of the vat is drawn off at the rate of 3 liters/min. Which of the initial value problems below best describes the quantity of sodium hydroxide, $Q(t)$, that would be in the vat at time t, $0 < t < 80$?

(a) $Q' = 10 - \dfrac{3}{80-t}Q, \qquad Q(0) = 160.$

(b) $Q' = 10 - \dfrac{3}{100}Q, \qquad Q(0) = 160.$

(c) $Q' = 10 - \dfrac{3}{80+t}Q, \qquad Q(0) = 200.$

(d) $Q' = 10 - \dfrac{3}{100-t}Q, \qquad Q(0) = 200.$

Units: liters, grams, min

$Q(t)$ – the amount of sodium hydroxide (grams) in the vat.

Initial condition

$Q(0) = 80 \cdot 2 = 160$

Dif. equation:

$$Q'(t) = r_{in} c_{in} - r_{out} c_{out} = 2\cdot 5 - 3\frac{Q(t)}{80-t}$$

$r_{in} = 2 \frac{\text{liters}}{\text{min}}$

$c_{in} = 5 \frac{\text{grams}}{\text{liter}}$

$r_{out} = 3 \frac{\text{liters}}{\text{min}}$

$c_{out} = \dfrac{Q(t)}{W(t)} = \dfrac{Q(t)}{80-t}$

Note $r_{in} \neq r_{out}$

$$\begin{cases} W'(t) = r_{in} - r_{out} = 2 - 3 = -1 \\ W(0) = 80 \end{cases} \qquad W(t) = 80 - t$$

8. (5 points) Consider the autonomous equation

$$y' = y^3 - 12y^2 + 20y = y(y-2)(y-10).$$

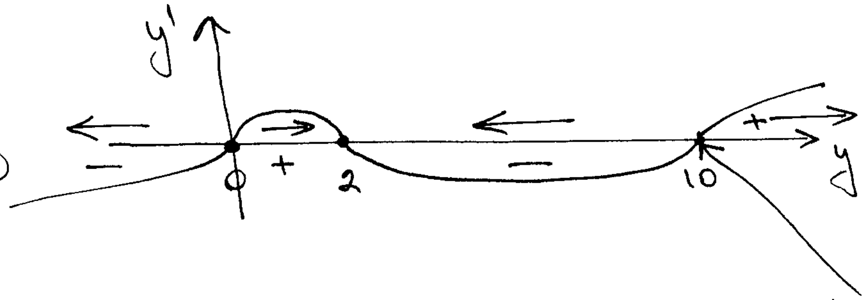
Suppose $y(\frac{\pi}{2}) = 10$. What is $\lim_{t\to\infty} y(t)$?

(a) 0

(b) 2

(c) 10

(d) ∞



$y(t) = 10$ is equilibrium solution

initial value of y (at $t = \pi/2$)

9. (5 points) Solve the following initial value problem

$$y'' + 3y' + 2y = 0, \qquad y(1) = 1, \qquad y'(1) = 0.$$

(a) $2e^{-(t-1)} - e^{-2(t-1)}$

(b) $2e^{-(t+1)} - e^{-2(t+1)}$

(c) $e^{-t} - e^{-2t}$

(d) $3e^{2t} - 2e^{3t}$

Characteristic equation

$$r^2 + 3r + 2 = 0$$

$$r_{1,2} = \frac{-3 \pm \sqrt{3^2 - 4\cdot 2}}{2} = \frac{-3 \pm 1}{2} = \begin{cases} -2 \\ -1 \end{cases}$$

$$r^2 + 3r + 2 = (r+2)(r+1)$$

General solution

$$y(t) = C_1 e^{-t} + C_2 e^{-2t}$$

Compute C_1 and C_2 from initial conditions

$$\begin{cases} y(1) = C_1 e^{-1} + C_2 e^{-2} = 1 \\ y'(1) = -C_1 e^{-1} - 2C_2 e^{-2} = 0 \end{cases}$$

$\Rightarrow C_1 e^{-1} - \frac{e}{2} C_1 e^{-2} = C_1 e^{-1}\left(1 - \frac{1}{2}\right) = \frac{C_1 e^{-1}}{2} = 1$

$\Rightarrow C_2 = -\frac{1}{2}\frac{e^{-1}C_1}{e^{-2}} = -\frac{e}{2}C_1$

$$\boxed{\begin{array}{l} C_1 = 2e \\ C_2 = -e^2 \end{array}}$$

$$y(t) = 2e^{+1}e^{-t} - e^{+2}e^{-2t} = 2e^{-(t-1)} - e^{-2(t-1)}$$

10. (5 points) Given that $y_1 = t^2$ and $y_2 = \ln t$ are two solutions of a certain second order homogeneous linear equation, $y'' + p(t)y' + q(t)y = 0$. All of the functions below are also solutions of the equation, EXCEPT

(a) 0. $= 0 \cdot y_1 + 0 \cdot y_2$

(b) $-3\ln t$. $= 0 \cdot y_1 - 3 \cdot y_2$

(c) $2t^2 \ln t$.

(d) $t^2 - \ln t$. $= y_1 - y_2$

11. (5 points) Which of the equations below has $y(t) = 5e^{-3t}$ as a particular solution?

	Characteristic equations	
(a) $y'' + 9y = 0$	$r^2 + 9 = 0$	not a root
(b) $y'' - 6y' + 9y = 0$	$r^2 - 6r + 9 = 0$	not a root
(c) $y'' + y' - 6y = 0$	$r^2 + r - 6 = 0$	root
(d) $y'' - 2y' - 3y = 0$	$r^2 - 2r - 3 = 0$	not a root

$$y(t) = e^{-3t} \xrightarrow{\text{root}} r = -3$$

12. (12 points) Consider the nonhomogeneous second order linear equation of the form

$$y'' + 2y' + y = g(t).$$

(a) (3 points) Find $y_c(t)$, the solution of its corresponding homogeneous equation.

$y'' + 2y' + y = 0 \xrightarrow{\text{Char. equation}} r^2 + 2r + 1 = 0$

$(r+1)^2 = 0$

$r_{1,2} = -1, -1$

$\boxed{y_c(t) = (At + B)e^{-t}}$

For each of the parts (b) through (d), choose from the list below the function that is the most suitable choice of the **form** of particular solution Y that you would use to solve the given equation using the Method of Undetermined Coefficients. **DO NOT ATTEMPT TO SOLVE THE COEFFICIENTS.**

A.	$(At + B)e^{-t}$
B.	$(At^3 + Bt^2)e^{-t}$
C.	$Ae^{-t}\cos(t) + Be^{-t}\sin(t)$
D.	$Ae^{-t}\sin(t)$
E.	$(At^2 + Bt + C)e^t\cos(2t) + (Dt^2 + Et + F)e^t\sin(2t)$
F.	$At^2e^t\cos(2t) + Bt^2e^t\sin(2t)$
G.	$(At^2 + Bt)e^{-t}$
H.	$At^3e^t\cos(2t) + Bt^3e^t\sin(2t)$
I.	$At^2e^t\cos(2t)$

(b) (3 points) $y'' + 2y' + y = e^{-t}\sin t$

$e^{-t}(C_1 \sin t + C_2 \cos t)$

$\boxed{C}$

(c) (3 points) $y'' + 2y' + y = 2te^{-t}$

$\underbrace{t^2}_{\text{see } y_c}(C_1 t + C_2)e^{-t}$

$\boxed{B}$

(d) (3 points) $y'' + 2y' + y = -6t^2e^t\cos(2t)$

$(C_1 t^2 + C_2 t + C_3)e^t(C_4\cos(2t) + C_5\sin(2t))$

$\boxed{E}$

13. (9 points) A mass-spring system is described by the equation

$$u'' + 6u' + k\,u = 0.$$

(a) (3 points) Find the value(s) of k that would make the system critically damped.

Characteristic equation

$r^2 + 6r + K = 0$ $\qquad r_{1,2} = \dfrac{-6 \pm \sqrt{6^2 - 4K}}{2}$

Critically dumped when

$\sqrt{6^2 - 4K} = 0 \qquad 36 = 4K \Rightarrow \boxed{K = 9}$

(b) (3 points) If $k = 25$, what is the quasi-frequency of this mass-spring system?

$r_{1,2} = \dfrac{-6 \pm \sqrt{36 - 100}}{2} = -3 \pm 4i$

quasi-frequency $\boxed{\mu = 4}$

(c) (3 points) True or false: If $k = 6$, then there are some nonzero solutions of this mass-spring system that will cross the equilibrium position more than ten times.

False.

$\underline{K = 6} < 9$ corresponds to overdumped system.

A solution may cross equilibrium position at most 1 time.

14. (10 points) Solve the initial value problem:

$$ty' + 2y = \cos t, \qquad y(\pi) = 0.$$

Standard form

$$y' + \frac{2}{t}y = \frac{\cos t}{t} \quad \Big| \cdot \mu(t)$$

integrating factor

$$\mu'(t) = \frac{2}{t}\mu(t)$$

$$\underline{\mu(t) = t^2}$$

$$(t^2 y)' = \frac{\cos t}{t} t^2 = t\cos(t)$$

$$t^2 y(t) = \int t\cos(t)\,dt = \int t\,(\sin(t))'\,dt$$

integration by parts

$$= t\sin(t) - \int \sin(t) = t\sin(t) + \cos(t) + C$$

$$y(t) = \frac{\sin(t)}{t} + \frac{\cos(t)}{t^2} + \frac{C}{t^2}$$

Find C from initial conditions

$$y(\pi) = \frac{-1}{\pi^2} + \frac{C}{\pi^2} = 0 \implies \boxed{C = 1}$$

$$\boxed{y(t) = \frac{\sin(t)}{t} + \frac{\cos(t)}{t^2} + \frac{1}{t^2}}$$

15. (10 points) Given that $y_1(t) = t^3$ is a known solution of the second order linear differential equation

$$t^2y'' - 5ty' + 9y = 0, \qquad t > 0.$$

Find the general solution of the equation.

Standard form

$$y'' + \frac{-5t}{t^2}y' + \frac{9}{t^2}y = 0$$

$$y'' + \frac{-5}{t}y' + \frac{9}{t^2}y = 0$$

Abel's Thm

$$W(y_1, y_2) = \det\begin{vmatrix} t^3 & y_2 \\ 3t^2 & y_2' \end{vmatrix} = t^3 y_2' - 3t^2 y_2 = C_1 e^{-\int \frac{-5}{t}dt} =$$

$$= C_1 e^{5\int \frac{1}{t}dt} = C_1 e^{5\ln t} = C_1 t^5$$

Equation for $y_2(t)$:

$$t^3 y_2' - 3t^2 = C_1 t^5$$

In the standard form

$$y_2' + \frac{-3}{t}y_2 = C_1 t^2 \quad \Big| \cdot \mu$$

$$\mu'(t) = \frac{-3}{t}\mu(t) \implies \mu(t) = t^{-3}$$

$$(t^{-3}y_2)' = C_1 t^2 t^{-3} = C t^{-1}$$

$$t^{-3}y_2 = C_1\int t^{-1} = C_1 \ln t + C_2$$

$$\boxed{y_2 = C_1 t^3 \ln t + C_2 t^3} \longleftarrow \text{General solution}$$