

which is the formal solution of the given problem. It is also possible to write y in the form

$$y = \begin{cases} 0, & t < 5, \\ \frac{2}{\sqrt{15}} e^{-(t-5)/4} \sin \frac{\sqrt{15}}{4} (t-5), & t \geq 5. \end{cases} \quad (22)$$

The graph of Eq. (22) is shown in Figure 6.5.3. Since the initial conditions at $t = 0$ are homogeneous and there is no external excitation until $t = 5$, there is no response in the interval $0 < t < 5$. The impulse at $t = 5$ produces a decaying oscillation that persists indefinitely. The response is continuous at $t = 5$ despite the singularity in the forcing function at that point. However, the first derivative of the solution has a jump discontinuity at $t = 5$, and the second derivative has an infinite discontinuity there. This is required by the differential equation (17), since a singularity on one side of the equation must be balanced by a corresponding singularity on the other side.

In dealing with problems with impulsive forcing the use of the delta function usually simplifies the mathematical calculations, often quite significantly. However, if the actual excitation extends over a short, but nonzero, time interval, then an error will be introduced by modeling the excitation as taking place instantaneously. This error may be negligible, but in a practical problem it should not be dismissed without consideration. In Problem 16 you are asked to investigate this issue for a simple harmonic oscillator.

PROBLEMS

In each of Problems 1 through 12:

- (a) Find the solution of the given initial value problem.
- (b) Draw a graph of the solution.

1. $y'' + 2y' + 2y = \delta(t - \pi); \quad y(0) = 1, \quad y'(0) = 0$
2. $y'' + 4y = \delta(t - \pi) - \delta(t - 2\pi); \quad y(0) = 0, \quad y'(0) = 0$
3. $y'' + 3y' + 2y = \delta(t - 5) + u_{10}(t); \quad y(0) = 0, \quad y'(0) = 1/2$
4. $y'' - y = -20\delta(t - 3); \quad y(0) = 1, \quad y'(0) = 0$
5. $y'' + 2y' + 3y = \sin t + \delta(t - 3\pi); \quad y(0) = 0, \quad y'(0) = 0$
6. $y'' + 4y = \delta(t - 4\pi); \quad y(0) = 1/2, \quad y'(0) = 0$
7. $y'' + y = \delta(t - 2\pi) \cos t; \quad y(0) = 0, \quad y'(0) = 1$
8. $y'' + 4y = 2\delta(t - \pi/4); \quad y(0) = 0, \quad y'(0) = 0$
9. $y'' + y = u_{\pi/2}(t) + 3\delta(t - 3\pi/2) - u_{2\pi}(t); \quad y(0) = 0, \quad y'(0) = 0$
10. $2y'' + y' + 4y = \delta(t - \pi/6) \sin t; \quad y(0) = 0, \quad y'(0) = 0$
11. $y'' + 2y' + 2y = \cos t + \delta(t - \pi/2); \quad y(0) = 0, \quad y'(0) = 0$
12. $y^{(4)} - y = \delta(t - 1); \quad y(0) = 0, \quad y'(0) = 0, \quad y''(0) = 0, \quad y'''(0) = 0$
13. Consider again the system in Example 1 of this section, in which an oscillation is excited by a unit impulse at $t = 5$. Suppose that it is desired to bring the system to rest again after exactly one cycle—that is, when the response first returns to equilibrium moving in the positive direction.
 - (a) Determine the impulse $k\delta(t - t_0)$ that should be applied to the system in order to accomplish this objective. Note that k is the magnitude of the impulse and t_0 is the time of its application.

(b) Solve the resulting initial value problem and plot its solution to confirm that it behaves in the specified manner.

14. Consider the initial value problem

$$y'' + \gamma y' + y = \delta(t - 1), \quad y(0) = 0, \quad y'(0) = 0,$$

where γ is the damping coefficient (or resistance).

- (a) Let $\gamma = \frac{1}{2}$. Find the solution of the initial value problem and plot its graph.
 (b) Find the time t_1 at which the solution attains its maximum value. Also find the maximum value y_1 of the solution.
 (c) Let $\gamma = \frac{1}{4}$ and repeat parts (a) and (b).
 (d) Determine how t_1 and y_1 vary as γ decreases. What are the values of t_1 and y_1 when $\gamma = 0$?
15. Consider the initial value problem

$$y'' + \gamma y' + y = k\delta(t - 1), \quad y(0) = 0, \quad y'(0) = 0,$$

where k is the magnitude of an impulse at $t = 1$ and γ is the damping coefficient (or resistance).

- (a) Let $\gamma = \frac{1}{2}$. Find the value of k for which the response has a peak value of 2; call this value k_1 .
 (b) Repeat part (a) for $\gamma = \frac{1}{4}$.
 (c) Determine how k_1 varies as γ decreases. What is the value of k_1 when $\gamma = 0$?
16. Consider the initial value problem

$$y'' + y = f_k(t), \quad y(0) = 0, \quad y'(0) = 0,$$

where $f_k(t) = [u_{4-k}(t) - u_{4+k}(t)]/2k$ with $0 < k \leq 1$.

- (a) Find the solution $y = \phi(t, k)$ of the initial value problem.
 (b) Calculate $\lim_{k \rightarrow 0} \phi(t, k)$ from the solution found in part (a).
 (c) Observe that $\lim_{k \rightarrow 0} f_k(t) = \delta(t - 4)$. Find the solution $\phi_0(t)$ of the given initial value problem with $f_k(t)$ replaced by $\delta(t - 4)$. Is it true that $\phi_0(t) = \lim_{k \rightarrow 0} \phi(t, k)$?
 (d) Plot $\phi(t, 1/2)$, $\phi(t, 1/4)$, and $\phi_0(t)$ on the same axes. Describe the relation between $\phi(t, k)$ and $\phi_0(t)$.

Problems 17 through 22 deal with the effect of a sequence of impulses on an undamped oscillator. Suppose that

$$y'' + y = f(t), \quad y(0) = 0, \quad y'(0) = 0.$$

For each of the following choices for $f(t)$:

- (a) Try to predict the nature of the solution without solving the problem.
 (b) Test your prediction by finding the solution and drawing its graph.
 (c) Determine what happens after the sequence of impulses ends.
17. $f(t) = \sum_{k=1}^{20} \delta(t - k\pi)$
18. $f(t) = \sum_{k=1}^{20} (-1)^{k+1} \delta(t - k\pi)$
19. $f(t) = \sum_{k=1}^{20} \delta(t - k\pi/2)$
20. $f(t) = \sum_{k=1}^{20} (-1)^{k+1} \delta(t - k\pi/2)$

21. $f(t) = \sum_{k=1}^{15} \delta[t - (2k - 1)\pi]$ 22. $f(t) = \sum_{k=1}^{40} (-1)^{k+1} \delta(t - 11k/4)$
23. The position of a certain lightly damped oscillator satisfies the initial value problem

$$y'' + 0.1y' + y = \sum_{k=1}^{20} (-1)^{k+1} \delta(t - k\pi), \quad y(0) = 0, \quad y'(0) = 0.$$

Observe that, except for the damping term, this problem is the same as Problem 18.

- (a) Try to predict the nature of the solution without solving the problem.
 (b) Test your prediction by finding the solution and drawing its graph.
 (c) Determine what happens after the sequence of impulses ends.
24. Proceed as in Problem 23 for the oscillator satisfying

$$y'' + 0.1y' + y = \sum_{k=1}^{15} \delta[t - (2k - 1)\pi], \quad y(0) = 0, \quad y'(0) = 0.$$

Observe that, except for the damping term, this problem is the same as Problem 21.

25. (a) By the method of variation of parameters, show that the solution of the initial value problem

$$y'' + 2y' + 2y = f(t); \quad y(0) = 0, \quad y'(0) = 0$$

is

$$y = \int_0^t e^{-(t-\tau)} f(\tau) \sin(t - \tau) d\tau.$$

- (b) Show that if $f(t) = \delta(t - \pi)$, then the solution of part (a) reduces to

$$y = u_\pi(t) e^{-(t-\pi)} \sin(t - \pi).$$

- (c) Use a Laplace transform to solve the given initial value problem with $f(t) = \delta(t - \pi)$ and confirm that the solution agrees with the result of part (b).

6.6 The Convolution Integral

Sometimes it is possible to identify a Laplace transform $H(s)$ as the product of two other transforms $F(s)$ and $G(s)$, the latter transforms corresponding to known functions f and g , respectively. In this event, we might anticipate that $H(s)$ would be the transform of the product of f and g . However, this is not the case; in other words, the Laplace transform cannot be commuted with ordinary multiplication. On the other hand, if an appropriately defined “generalized product” is introduced, then the situation changes, as stated in the following theorem.

Theorem 6.6.1 If $F(s) = \mathcal{L}\{f(t)\}$ and $G(s) = \mathcal{L}\{g(t)\}$ both exist for $s > a \geq 0$, then

$$H(s) = F(s)G(s) = \mathcal{L}\{h(t)\}, \quad s > a, \quad (1)$$