Theorem 6.3.2 If $F(s) = \mathcal{L}\{f(t)\}$ exists for $s > a \geq 0$, and if c is a constant, then

$$\mathcal{L}\{e^{ct}f(t)\} = F(s-c), \qquad s > a + c. \tag{7}$$

Conversely, if $f(t) = \mathcal{L}^{-1}\{F(s)\}$, then

$$e^{ct}f(t) = \mathcal{L}^{-1}\{F(s-c)\}. \tag{8}$$

According to Theorem 6.3.2, multiplication of $f(t)$ by e^{ct} results in a translation of the transform $F(s)$ a distance c in the positive s direction, and conversely. To prove this theorem, we evaluate $\mathcal{L}\{e^{ct}f(t)\}$. Thus

$$\mathcal{L}\{e^{ct}f(t)\} = \int_0^\infty e^{-st}e^{ct}f(t)\,dt = \int_0^\infty e^{-(s-c)t}f(t)\,dt$$
$$= F(s-c),$$

which is Eq. (7). The restriction $s > a + c$ follows from the observation that, according to hypothesis (ii) of Theorem 6.1.2, $|f(t)| \leq Ke^{at}$; hence $|e^{ct}f(t)| \leq Ke^{(a+c)t}$. Equation (8) is obtained by taking the inverse transform of Eq. (7), and the proof is complete.

The principal application of Theorem 6.3.2 is in the evaluation of certain inverse transforms, as illustrated by Example 5.

EXAMPLE 5

Find the inverse transform of

$$G(s) = \frac{1}{s^2 - 4s + 5}.$$

By completing the square in the denominator, we can write

$$G(s) = \frac{1}{(s-2)^2 + 1} = F(s-2),$$

where $F(s) = (s^2+1)^{-1}$. Since $\mathcal{L}^{-1}\{F(s)\} = \sin t$, it follows from Theorem 6.3.2 that

$$g(t) = \mathcal{L}^{-1}\{G(s)\} = e^{2t}\sin t.$$

The results of this section are often useful in solving differential equations, particularly those that have discontinuous forcing functions. The next section is devoted to examples illustrating this fact.

PROBLEMS

In each of Problems 1 through 6 sketch the graph of the given function on the interval $t \geq 0$.

1. $g(t) = u_1(t) + 2u_3(t) - 6u_4(t)$
2. $g(t) = (t-3)u_2(t) - (t-2)u_3(t)$
3. $g(t) = f(t-\pi)u_\pi(t)$, where $f(t) = t^2$
4. $g(t) = f(t-3)u_3(t)$, where $f(t) = \sin t$
5. $g(t) = f(t-1)u_2(t)$, where $f(t) = 2t$
6. $g(t) = (t-1)u_1(t) - 2(t-2)u_2(t) + (t-3)u_3(t)$

In each of Problems 7 through 12:
(a) Sketch the graph of the given function.
(b) Express $f(t)$ in terms of the unit step function $u_c(t)$.

7. $f(t) = \begin{cases} 0, & 0 \le t < 3, \\ -2, & 3 \le t < 5, \\ 2, & 5 \le t < 7, \\ 1, & t \ge 7. \end{cases}$

8. $f(t) = \begin{cases} 1, & 0 \le t < 1, \\ -1, & 1 \le t < 2, \\ 1, & 2 \le t < 3, \\ -1, & 3 \le t < 4, \\ 0, & t \ge 4. \end{cases}$

9. $f(t) = \begin{cases} 1, & 0 \le t < 2, \\ e^{-(t-2)}, & t \ge 2. \end{cases}$

10. $f(t) = \begin{cases} t^2, & 0 \le t < 2, \\ 1, & t \ge 2. \end{cases}$

11. $f(t) = \begin{cases} t, & 0 \le t < 1, \\ t-1, & 1 \le t < 2, \\ t-2, & 2 \le t < 3, \\ 0, & t \ge 3. \end{cases}$

12. $f(t) = \begin{cases} t, & 0 \le t < 2, \\ 2, & 2 \le t < 5, \\ 7-t, & 5 \le t < 7, \\ 0, & t \ge 7. \end{cases}$

In each of Problems 13 through 18 find the Laplace transform of the given function.

13. $f(t) = \begin{cases} 0, & t < 2 \\ (t-2)^2, & t \ge 2 \end{cases}$

14. $f(t) = \begin{cases} 0, & t < 1 \\ t^2 - 2t + 2, & t \ge 1 \end{cases}$

15. $f(t) = \begin{cases} 0, & t < \pi \\ t - \pi, & \pi \le t < 2\pi \\ 0, & t \ge 2\pi \end{cases}$

16. $f(t) = u_1(t) + 2u_3(t) - 6u_4(t)$

17. $f(t) = (t-3)u_2(t) - (t-2)u_3(t)$

18. $f(t) = t - u_1(t)(t-1), \qquad t \ge 0$

In each of Problems 19 through 24 find the inverse Laplace transform of the given function.

19. $F(s) = \dfrac{3!}{(s-2)^4}$

20. $F(s) = \dfrac{e^{-2s}}{s^2 + s - 2}$

21. $F(s) = \dfrac{2(s-1)e^{-2s}}{s^2 - 2s + 2}$

22. $F(s) = \dfrac{2e^{-2s}}{s^2 - 4}$

23. $F(s) = \dfrac{(s-2)e^{-s}}{s^2 - 4s + 3}$

24. $F(s) = \dfrac{e^{-s} + e^{-2s} - e^{-3s} - e^{-4s}}{s}$

25. Suppose that $F(s) = \mathcal{L}\{f(t)\}$ exists for $s > a \ge 0$.
   (a) Show that if c is a positive constant, then

$$\mathcal{L}\{f(ct)\} = \frac{1}{c}F\left(\frac{s}{c}\right), \qquad s > ca.$$

   (b) Show that if k is a positive constant, then

$$\mathcal{L}^{-1}\{F(ks)\} = \frac{1}{k}f\left(\frac{t}{k}\right).$$

   (c) Show that if a and b are constants with $a > 0$, then

$$\mathcal{L}^{-1}\{F(as+b)\} = \frac{1}{a}e^{-bt/a}f\left(\frac{t}{a}\right).$$

In each of Problems 26 through 29 use the results of Problem 25 to find the inverse Laplace transform of the given function.

26. $F(s) = \dfrac{2^{n+1}n!}{s^{n+1}}$

27. $F(s) = \dfrac{2s+1}{4s^2+4s+5}$

28. $F(s) = \dfrac{1}{9s^2-12s+3}$

29. $F(s) = \dfrac{e^2e^{-4s}}{2s-1}$

In each of Problems 30 through 33 find the Laplace transform of the given function. In Problem 33 assume that term-by-term integration of the infinite series is permissible.

30. $f(t) = \begin{cases} 1, & 0 \le t < 1 \\ 0, & t \ge 1 \end{cases}$

31. $f(t) = \begin{cases} 1, & 0 \le t < 1 \\ 0, & 1 \le t < 2 \\ 1, & 2 \le t < 3 \\ 0, & t \ge 3 \end{cases}$

32. $f(t) = 1 - u_1(t) + \cdots + u_{2n}(t) - u_{2n+1}(t) = 1 + \sum_{k=1}^{2n+1}(-1)^k u_k(t)$

33. $f(t) = 1 + \sum_{k=1}^{\infty}(-1)^k u_k(t)$. See Figure 6.3.7.

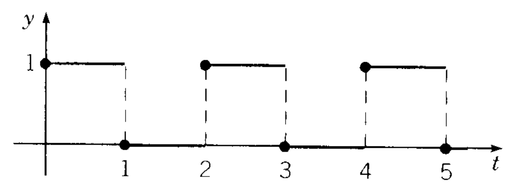

FIGURE 6.3.7 A square wave.

34. Let f satisfy $f(t+T) = f(t)$ for all $t \ge 0$ and for some fixed positive number T; f is said to be periodic with period T on $0 \le t < \infty$. Show that

$$\mathcal{L}\{f(t)\} = \frac{\int_0^T e^{-st} f(t)\,dt}{1 - e^{-sT}}.$$

In each of Problems 35 through 38, use the result of Problem 34 to find the Laplace transform of the given function.

35. $f(t) = \begin{cases} 1, & 0 \le t < 1, \\ 0, & 1 \le t < 2; \end{cases}$

$f(t+2) = f(t)$.

Compare with Problem 33.

36. $f(t) = \begin{cases} 1, & 0 \le t < 1, \\ -1, & 1 \le t < 2; \end{cases}$

$f(t+2) = f(t)$.

See Figure 6.3.8.

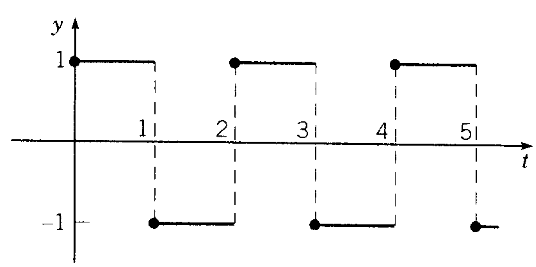

FIGURE 6.3.8 A square wave.

37. $f(t) = t, \quad 0 \le t < 1;$
$f(t+1) = f(t).$
See Figure 6.3.9.

38. $f(t) = \sin t, \quad 0 \le t < \pi;$
$f(t+\pi) = f(t).$
See Figure 6.3.10.

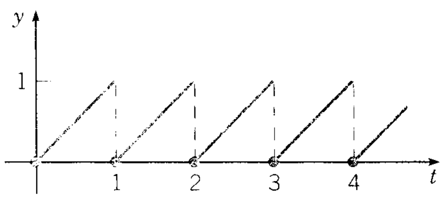

FIGURE 6.3.9 A sawtooth wave.

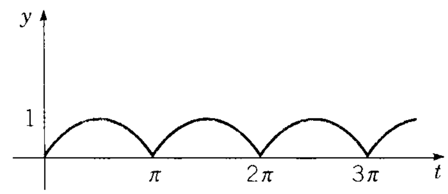

FIGURE 6.3.10 A rectified sine wave.

39. (a) If $f(t) = 1 - u_1(t)$, find $\mathcal{L}\{f(t)\}$; compare with Problem 30. Sketch the graph of $y = f(t)$.

(b) Let $g(t) = \int_0^t f(\xi)\,d\xi$, where the function f is defined in part (a). Sketch the graph of $y = g(t)$ and find $\mathcal{L}\{g(t)\}$.

(c) Let $h(t) = g(t) - u_1(t)g(t-1)$, where g is defined in part (b). Sketch the graph of $y = h(t)$ and find $\mathcal{L}\{h(t)\}$.

40. Consider the function p defined by

$$p(t) = \begin{cases} t, & 0 \le t < 1, \\ 2 - t, & 1 \le t < 2; \end{cases} \qquad p(t+2) = p(t).$$

(a) Sketch the graph of $y = p(t)$.

(b) Find $\mathcal{L}\{p(t)\}$ by noting that p is the periodic extension of the function h in Problem 39(c) and then using the result of Problem 34.

(c) Find $\mathcal{L}\{p(t)\}$ by noting that

$$p(t) = \int_0^t f(t)\,dt,$$

where f is the function in Problem 36, and then using Theorem 6.2.1.

6.4 Differential Equations with Discontinuous Forcing Functions

In this section we turn our attention to some examples in which the nonhomogeneous term, or forcing function, is discontinuous.

EXAMPLE 1

Find the solution of the differential equation

$$2y'' + y' + 2y = g(t), \tag{1}$$

where

$$g(t) = u_5(t) - u_{20}(t) = \begin{cases} 1, & 5 \le t < 20, \\ 0, & 0 \le t < 5 \quad \text{and} \quad t \ge 20. \end{cases} \tag{2}$$