(1)

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 \\ 2 & 3 & 1 & h \end{array}\right] \xrightarrow[R_3 \leftarrow R_3 - 2R_1]{R_2 \leftarrow R_2 - R_1} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & h-2 \end{array}\right] \longrightarrow$$

$$\xrightarrow{R_3 \leftarrow R_3 - R_2} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & h-2 \end{array}\right]$$

This system is consistent for all values of h.

(C)

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(2)

$$\left[\begin{array}{cc|c} 1 & -2 & -1 \\ 2 & -3 & 5 \\ -1 & 3 & h \end{array}\right] \xrightarrow[R_3 \leftarrow R_3 + R_1]{R_2 \leftarrow R_2 - 2R_1} \left[\begin{array}{cc|c} 1 & -2 & -1 \\ 0 & 1 & 7 \\ 0 & 1 & h-1 \end{array}\right] \longrightarrow$$

$$\xrightarrow{R_3 \leftarrow R_3 - R_2} \left[\begin{array}{cc|c} 1 & -2 & -1 \\ 0 & 1 & 7 \\ 0 & 0 & h-8 \end{array}\right]$$

This system is consistent (hence lines have a common point) if $h - 8 = 0 \iff h = 8$

(a)

(3)

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$ echelon form

$$B = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 2 \\ 0 & 0 & 2 & 0 \end{bmatrix}$$ echelon form

$$C = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$ reduced echelon form

(c)

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(4)

$$\begin{bmatrix} 5 & 4 & 3 \\ 1 & 2 & 3 \\ 1 & 1 & 1 \end{bmatrix} \xrightarrow{R_1 \Leftrightarrow R_3} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 5 & 4 & 3 \end{bmatrix} \xrightarrow[R_3 \leftarrow R_3 - 5R_1]{R_2 \leftarrow R_2 - R_1}$$

$$\longrightarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & -1 & -2 \end{bmatrix} \xrightarrow{R_3 \leftarrow R_3 + R_2} \begin{bmatrix} \boxed{1} & 1 & 1 \\ 0 & \boxed{1} & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{array}{ccc} \uparrow & \uparrow & \end{array}$$

Columns 1 and 2 are pivot columns.

(b)

5 $\vec{c} = 3\left(\vec{a} + \frac{1}{2}\vec{b}\right) \Longrightarrow$

$\text{Span}\{\vec{a}, \vec{b}, \vec{c}\} = \text{Span}\{\underline{\vec{a}}, \underline{\vec{b}}\}$.

$\vec{a}$ and $\vec{b}$ are linearly independent.
Hence, it is a <u>plane</u>.

c

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6 $\vec{b} = c_1\vec{a}_1 + c_2\vec{a}_2 + \ldots + c_n\vec{a}_n \Longrightarrow$

$\vec{b} \in \text{Span}\{\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_n\}$.

a

---

7 $A \sim \begin{bmatrix} \square & * & * \\ 0 & \square & * \\ 0 & 0 & \square \end{bmatrix}, \quad B \sim \begin{bmatrix} ? & ? & ? & ? \\ 0 & ? & ? & ? \end{bmatrix}$

B has a pivot position in every row $\Longrightarrow$

$B\vec{x} = \vec{b}$

has at <u>least one</u> solution for <u>every</u> $\vec{b}$.

c

(5)

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 2 & 3 & 4 & 0 \\ 3 & 4 & 5 & 0 \end{array}\right] \xrightarrow[R_3 \leftarrow R_3 - 3R_1]{R_2 \leftarrow R_2 - 2R_1} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & -1 & -2 & 0 \\ 0 & -2 & -4 & 0 \end{array}\right] \xrightarrow{R_3 \leftarrow R_3 + 2R_2}$$

$$\rightarrow \left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & -1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{R_1 \leftarrow R_1 + 2R_2} \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & -1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

free

$$\begin{cases} x_1 = x_3 \\ x_2 = -2x_3 \\ x_3 = \text{(free) anything} \end{cases}$$

To find the parametric vector form

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_3 \\ -2x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

Here x_3 is a parameter. We can write in place of x_3 any other letter as well.

(c)

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(9)

$\{v_1, v_5\} = \left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$ is the set of linearly independent vectors (they are not multiples of each other).

(d)

(10)

$$A = \begin{bmatrix} \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & \bullet \end{bmatrix} \sim \begin{bmatrix} \square & * & * & * & * \\ 0 & \square & * & * & * \\ 0 & 0 & \square & * & * \\ 0 & 0 & 0 & \square & * \\ 0 & 0 & 0 & 0 & \square \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

2 rows of zeros.

(c)

---

(11) Only $T(x) = -2x$ is linear.

For linear transformation we must have

(d) $T(cx) = cT(x) \Rightarrow \underline{T(\vec{0})} = T(0\cdot\vec{x}) = 0T(\vec{x}) \underline{= 0}$

---

(12) $A = \begin{bmatrix} T(\vec{e}_1) & T(\vec{e}_2) \end{bmatrix}$

$$T(\vec{e}_1) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}; \quad T(\vec{e}_2) = T\left(\tfrac{1}{2}\left(-(\vec{e}_1 - 2\vec{e}_2) + \vec{e}_1\right)\right) =$$

$$= -\tfrac{1}{2}T(\vec{e}_1 - 2\vec{e}_2) + \tfrac{1}{2}T(\vec{e}_1) = -\tfrac{1}{2}\begin{bmatrix} 3 \\ -5 \end{bmatrix} + \tfrac{1}{2}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix}$$

(d)

13 Columns of A are linearly dependent $\Rightarrow$

$\Rightarrow$ A is not one-to-one

$\text{Span}\{\vec{a}_1, \vec{a}_2, \vec{a}_3\} = \mathbb{R}^2 \Rightarrow$

$\Rightarrow$ A is onto.

(c)

14
$$\begin{bmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ 0 & 1 & 0 & 1 \\ \cdot & \cdot & \cdot & \cdot \end{bmatrix} \begin{bmatrix} \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot & 0 \end{bmatrix} = \begin{bmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & c_{34} \\ \cdot & \cdot & \cdot & \cdot \end{bmatrix}$$

$$c_{34} = 0 \cdot 1 + 1 \cdot 0 + 0 \cdot 1 + 1 \cdot 0 = 0$$

(b)

15
$$\begin{bmatrix} -1 & 2 \\ 1 & -2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ -2 & 3 & 1 \end{bmatrix} = \begin{bmatrix} \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{bmatrix} \quad -1 \cdot 1 + 2 \cdot 1 = 1$$

But we need $(AB)^T \Rightarrow$

$$\begin{bmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot \end{bmatrix}$$

(b)

(16) $(AB)^T = B^T A^T \neq A^T B^T$

in general

(d)

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(17) A is <u>not</u> invertible if its columns $\vec{a}_1, \vec{a}_2, \vec{a}_3, \vec{a}_4$ are linearly dependent.

$$\begin{array}{c|c|c|c} h & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 2 \end{array}$$

$\vec{a}_1$ may depend linearly only on $\vec{a}_3$.

This happens only when

$\underline{h=1}$

(c)

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(18) (a) $A \sim I$.

$[\,A \mid I\,] \sim [\,I \mid A^{-1}\,]$.

(19) $A = \begin{bmatrix}1&1&1\\1&2&1\end{bmatrix}$, Solving $A\vec{x}=\vec{0}$

$$\left[\begin{array}{ccc|c}1&1&1&0\\1&2&1&0\end{array}\right] \xrightarrow{R_2\leftarrow R_2-R_1} \left[\begin{array}{ccc|c}1&1&1&0\\0&1&0&0\end{array}\right] \xrightarrow{R_1\leftarrow R_1-R_2}$$

free

$$\longrightarrow \left[\begin{array}{ccc|c}1&0&1&0\\0&1&0&0\end{array}\right]$$

$$\begin{cases} x_1 = -x_3 \\ x_2 = 0 \end{cases} \qquad \vec{x} = \begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}-x_3\\0\\x_3\end{bmatrix} = x_3\underline{\begin{bmatrix}-1\\0\\1\end{bmatrix}}$$

The basis is $\left\{\begin{bmatrix}-1\\0\\1\end{bmatrix}\right\}$ (a)

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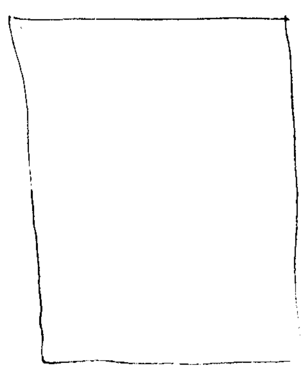
(20)

n

m



Col (A) is a subspace of $\mathbb{R}^m$ <u>not</u> $\mathbb{R}^n$

(C)