$\boxed{\text{Solutions}}$

Math 250
Section: 4

Quiz 3
Fall 2007

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Date: November 16, 2007

All work must be shown to receive full credit!

1. *(5 points)* Assuming that $\mathcal{L}\{f(t)\}(s) = F(s)$ compute the following:

$\mathcal{L}\{f^{(n)}(t)\}(s) =$ $s^n F(s) - s^{n-1} f(0) - \ldots - s f^{(n-2)}(0) - f^{(n-1)}(0).$

$\mathcal{L}\{e^{ct} f(t)\}(s) =$ $F(s-c)$

$\mathcal{L}\{t^n f(t)\}(s) =$ $(-1)^n F^{(n)}(s)$

$\mathcal{L}\{u_c(t) f(t-c)\}(s) =$ $e^{-cs} F(s)$

$\mathcal{L}\{\delta_c(t) f(t)\}(s) =$ $e^{-cs} f(c)$

2. *(5+10 points)* Using Laplace transform compute the solution to the following initial value problem:

$$\begin{cases} y'' + 2y' + 2y = h(t), \\ y(0) = 0, \\ y'(0) = 1. \end{cases}$$

Here

$$h(t) = \begin{cases} 0, & t < 1 \\ t-1, & t > 1. \end{cases}$$

$\underline{h(t) = u_1(t)\cdot(t-1)}$

(a) Assuming that $Y(s) = \mathcal{L}\{y(t)\}(s)$, solve for $Y(s)$.

$$\mathcal{L}\{y''+2y'+2y\}(s) = (s^2+2s+2)Y(s) - s\underbrace{\cancel{y(0)}}_{0} - \underbrace{y'(0)}_{1} - 2\underbrace{\cancel{y(0)}}_{0} =$$

$$= \underline{(s^2+2s+2)Y(s) - 1}.$$

$$\mathcal{L}\{u_1(t)(t-1)\}(s) = e^{-s}\mathcal{L}\{t\}(s) = \underline{e^{-s}\frac{1}{s^2}}.$$

$$\boxed{Y(s) = \frac{1}{s^2+2s+2} + \frac{1}{s^2(s^2+2s+2)}e^{-s}}$$

(b) Based on $Y(s)$ find $y(t)$.

$$\frac{1}{s^2+2s+2} = \frac{1}{(s+1)^2+1}.$$

$$\frac{1}{s^2(s^2+2s+2)} = \frac{As+B}{s^2} + \frac{Cs+D}{s^2+2s+2} = \frac{1}{2}\left[\frac{-s+1}{s^2} + \frac{s+1}{(s+1)^2+1}\right] =$$

$$= \frac{1}{2}\left[\frac{-1}{s} + \frac{1}{s^2} + \frac{(s+1)}{(s+1)^2+1}\right].$$

$s^3: \quad A+C=0 \qquad \boxed{C=\frac{1}{2}}$

$s^2: \quad 2A+B+D=0 \qquad \boxed{D=\frac{1}{2}}$

$s: \quad 2A+2B=0 \qquad \boxed{A=\frac{-1}{2}}$

$1: \quad 2B=1 \qquad \boxed{B=\frac{1}{2}}$

$$\mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2+1}\right\}(t) = e^{-t}\mathcal{L}^{-1}\left\{\frac{1}{s^2+1^2}\right\}(t) = \underline{e^{-t}\sin(t)}.$$

$$\mathcal{L}^{-1}\left\{\frac{1}{2}\left[\frac{-1}{s} + \frac{1}{s^2} + \frac{(s+1)}{(s+1)^2+1}\right]e^{-s}\right\}(t) = \frac{1}{2}u_1(t)\,\mathcal{L}^{-1}\left\{\frac{-1}{s} + \frac{1}{s^2} + \frac{(s+1)}{(s+1)^2+1}\right\}(t-1) =$$

$$= \frac{1}{2}u_1(t)\left[-1 + (t-1) + e^{-(t-1)}\mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\}(t-1)\right] =$$

$$= \frac{1}{2}u_1(t)\left[-1 + (t-1) + e^{-(t-1)}\cos(t-1)\right]$$

$$\boxed{y(t) = e^{-t}\sin t + \frac{1}{2}u_1(t)\left[-1 + (t-1) + e^{-(t-1)}\cos(t-1)\right]}$$