Sample problem

Find the general solution to the equation

$$y'' - 2y' - 3y = e^{3t}. \tag{1}$$

Step 1: Find the general solution to the homogeneous equation

$$y'' - 2y' - 3y = 0.$$

Characteristic equation

$$r^2 - 2r - 3 = 0, \qquad (r-3)(r+1) = 0, \qquad r_1 = 3, \quad r_2 = -1.$$

Two solutions, forming the fundamental set of solutions, are

$$y_1(t) = e^{3t}, \qquad y_2(t) = e^{-t}.$$

Step 2: Find a particular solution to equation (1). Note, that we cannot find a solution in the form

$$Y(t) = Ae^{3t}$$

since e^{3t} is a solution to the homogeneous equation.

What to do?

Use the technique similar to reduction of order.

Look for a solution in the form

$$Y(t) = v(t)y_1(t). \tag{2}$$

After substituting (2) in (1), note that all terms with $v(t)$ cancel

$$\begin{aligned} v''y_1 + v'\left[2y_1' - 2y_1\right] + v\underline{\left[y_1'' - 2y_1 - 3y_1\right]} &= e^{3t} \\ v''y_1 + v'\left[2y_1' - 2y_1\right] &= e^{3t} \\ v''e^{3t} + v'\left[2 \cdot 3e^{3t} - 2e^{3t}\right] &= e^{3t} \\ e^{3t}\left\{v'' + 4v'\right\} &= e^{3t} \end{aligned}$$

At this point the part that gave us trouble (e^{3t}) cancels

$$v'' + 4v' = 1, \qquad v' = 1/4, \qquad v = t/4.$$

Particular solution is $Y(t) = v(t)y_1(t) = \frac{t}{4}e^{3t}$.