Solutions

Math 250
Fall 2007
Final Exam

NAME: V. G.

ID No: ____________

SECTION: ____________

This exam contains 16 questions on 15 pages (including this title page). This exam is worth a total of 150 points. The exam is broken into two parts. There are eight multiple choice questions, each worth 5 points, and 8 partial credit problems. To receive full credit for a partial credit problem all work must be shown. When in doubt, fill in the details.

No notes, books or calculators may be used during the exam.

Please, Box Your Final Answer (when possible).

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Multiple Choice Section

1. *(5 points)* The autonomous equation $y' = -y^2 - y + 2$ has the following equilibrium solutions:

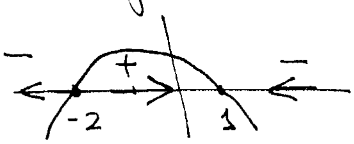
(a) $y = -2$ is an unstable solution and $y = 1$ is an asymptotically stable solution.

(b) $y = -2$ is an asymptotically stable solution and $y = 1$ is an unstable solution.

(c) $y = -2$ is an unstable solution and $y = 1$ is a semistable solution.

(d) $y = -2$ is a semistable solution and $y = 1$ is an asymptotically stable solution.

$$y' = 0 = -y^2 - y + 2 = -(y-1)(y+2)$$



$y = 1$ as. stable

$y = -2$ unstable

2. *(5 points)* On which of the following intervals is the unique solution of the initial value problem

$$(t-1)y' - (\ln t)y = \tan t, \quad y(2) = \frac{1}{2}$$

certain to exist?

$$y' - \frac{\ln t}{(t-1)} = \frac{\sin t}{\cos t\,(t-1)}$$

(a) $(0,\ 1)$

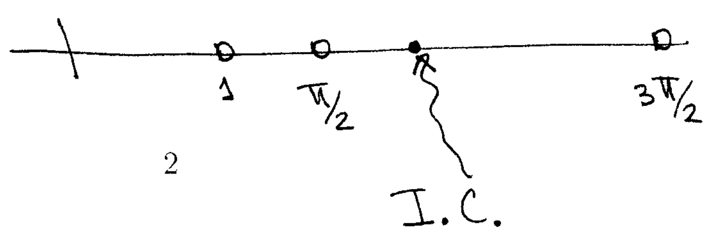
$t > 0, \quad t \neq 1, \quad t \neq \frac{\pi}{2} + \pi n$

(b) $(1,\ \frac{\pi}{2})$

Note $\pi \approx 3.14.. \Rightarrow \pi/2 > 1$

(c) $(1,\ \frac{3\pi}{2})$

(d) $(\frac{\pi}{2},\ \frac{3\pi}{2})$



3. *(5 points)* Which of the following differential equations has

$$y_1 = 3e^{2t} \quad \text{and} \quad y_2 = 5e^{-4t}$$

as two solutions?

Characteristic polynomial

$(r-2)(r+4) =$

$= r^2 + 2r - 8$

(a) $3y'' + 5y' + y = 0$

(b) $y'' + 2y' - 8y = 0$

(c) $y'' - 6y' + 8y = 0$

(d) $y'' - 2y' + 8y = 0$

4. *(5 points)* Find a suitable form of a particular solution to

$$y'' - 2y' + 2y = t(\sin t + 1).$$

The A, B, C, D, E, F below are constants.

(a) $At\sin t + Bt\cos t + Ct + D$

(b) $(At + B)(C\sin t + D\cos t + E)$

(c) $(At + B)\sin t + (Ct + D)\cos t + Et + F$

(d) $(At + B)t\sin t + (Ct + D)t\cos t + Et + F$

Hom. eq-n

$y'' - 2y' + 2y = 0$

$r^2 - 2r + 2 = 0$

$(r-1)^2 + 1 = 0$

$(r-1)^2 = -1, \Rightarrow r = 1 \pm i$

$y_1 = e^t \cos t$

$y_2 = e^t \sin t$

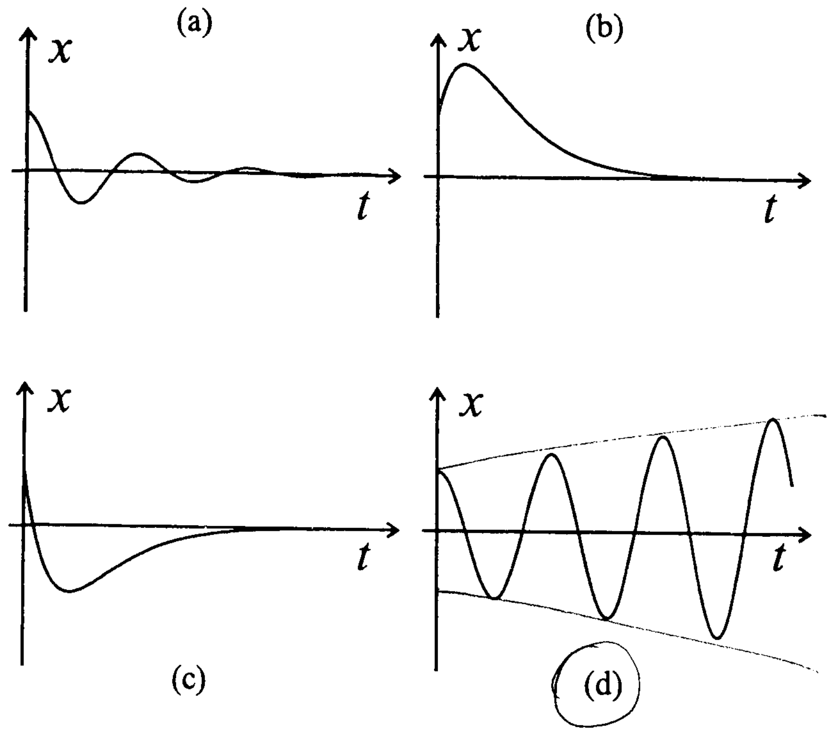
Initial guess

$(At + B)\sin t + (Ct + D)\cos t +$

$+ (Et + F)$

There are no overlaps with y_1, y_2.

5. *(5 points)* Which of the following graphs best depicts resonance?



6. *(5 points)* Which system corresponds to the differential equation

$$y'' + 2y' + 4y = \sin t \text{ ?}$$

(a) $\begin{cases} x_1' = x_2 \\ x_2' = -2x_1 - 4x_2 + \sin t \end{cases}$

(b) $\begin{cases} x_1' = x_2 \\ x_2' = -4x_1 - 2x_2 + \sin t \end{cases}$

(c) $\begin{cases} x_1' = x_2 \\ x_2' = 2x_1 + 4x_2 + \sin t \end{cases}$

(d) $\begin{cases} x_1' = x_2 \\ x_2' = -4x_1 - 2x_2 + \cos t \end{cases}$

$x_1 = y, \quad x_2 = x_1' = y'$

$x_2' + 2x_2 + 4x_1 = \sin t$

System

$\begin{cases} x_1' = x_2 \\ x_2' = -4x_1 - 2x_2 + \sin t \end{cases}$

$\lambda = 2$ has algebraic multiplicity 2
(geometric 1)

$(A - 2I)\vec{\xi} = 0$ | $(A - 2I)\vec{\eta} = \vec{\xi}$

$\left[\begin{array}{cc|c} 0 & 3 & 0 \\ 0 & 0 & 0 \end{array}\right]$ $\vec{\xi} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ | $\left[\begin{array}{cc|c} 0 & 3 & 1 \\ 0 & 0 & 0 \end{array}\right]$ $\vec{\eta} = \begin{bmatrix} 0 \\ 1/3 \end{bmatrix}$

7. *(5 points)* Find the general solution to the system

$$\mathbf{x}' = \begin{bmatrix} 2 & 3 \\ 0 & 2 \end{bmatrix} \mathbf{x}.$$

$\vec{X}_1(t) = e^{2t}\vec{\xi} = e^{2t}\begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$\vec{X}_2(t) = (\vec{\xi}t + \vec{\eta})e^{2t} = \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}t + \begin{bmatrix} 0 \\ 1/3 \end{bmatrix}\right)e^{2t}$

(a) $x_1 = c_1 e^{2t}$, $x_2 = c_2 t e^{2t}$, where c_1, c_2 are any constants.

(b) $x_1 = c_1 t e^{2t}$, $x_2 = c_2 e^{2t}$, where c_1, c_2 are any constants.

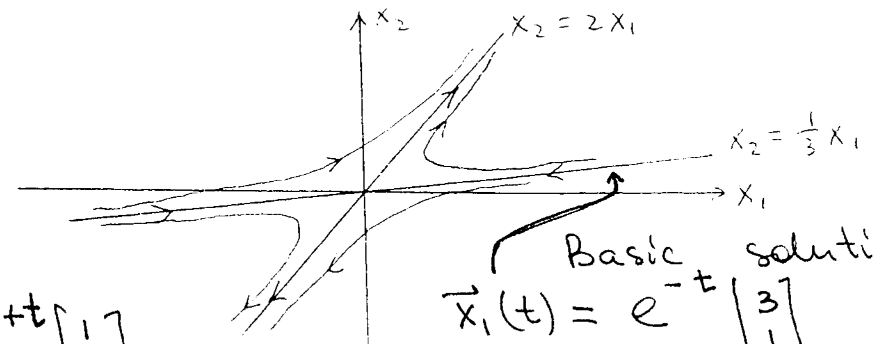
(c) $x_1 = c_1 e^{2t} + c_2 t e^{2t}$, $x_2 = \frac{1}{3} c_2 e^{2t}$, where c_1, c_2 are any constants.

(d) $x_1 = c_1 e^{2t} + c_2 t e^{2t}$, $x_2 = 3 c_2 t e^{2t}$, where c_1, c_2 are any constants.

General sol-n: $\vec{X}(t) = c_1\vec{X}_1 + c_2\vec{X}_2 = c_1\begin{bmatrix} e^{2t} \\ 0 \end{bmatrix} + c_2\begin{bmatrix} te^{2t} \\ \frac{1}{3}e^{2t} \end{bmatrix}$

$\boxed{x_1 = c_1 e^{2t} + c_2 t e^{2t}}$ $\boxed{x_2 = \frac{1}{3} c_2 e^{2t}}$

8. *(5 points)* Which of the following general solutions has the phase portrait given below?



Basic solutions:

$\vec{X}_1(t) = e^{-t}\begin{bmatrix} 3 \\ 1 \end{bmatrix}$

$\vec{X}_2(t) = e^{+t}\begin{bmatrix} 1 \\ 2 \end{bmatrix}$

(a) $\mathbf{x} = c_1 e^{-t}\begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2 e^{2t}\begin{pmatrix} 3 \\ 1 \end{pmatrix}$, where c_1, c_2 are any constants.

(b) $\mathbf{x} = c_1 e^{t}\begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2 e^{-t}\begin{pmatrix} 3 \\ 1 \end{pmatrix}$, where c_1, c_2 are any constants.

(c) $\mathbf{x} = c_1 e^{-t}\begin{pmatrix} 2 \\ 1 \end{pmatrix} + c_2 e^{2t}\begin{pmatrix} 1 \\ 3 \end{pmatrix}$, where c_1, c_2 are any constants.

(d) $\mathbf{x} = c_1 e^{t}\begin{pmatrix} 2 \\ 1 \end{pmatrix} + c_2 e^{-t}\begin{pmatrix} 1 \\ 3 \end{pmatrix}$, where c_1, c_2 are any constants.

Partial Credit Section

9. *(10 points)* Find the solution of the following initial value problem

$$ty' + 2(t-1)y = 3t^3e^t, \quad y(1) = -2e^{-2} + e.$$

Standard form:

$$y' + 2\left(1-\frac{1}{t}\right)y = 3t^2e^t$$

Use the integrating factor technique

$$\varphi y' + \underbrace{\varphi\, 2\left(1-\frac{1}{t}\right)}_{=\varphi'}y = 3t^2e^t\varphi$$

Solve $\varphi' = 2\left(1-\frac{1}{t}\right)\varphi \Rightarrow \boxed{\varphi = e^{2(t-\ln t)} = \frac{1}{t^2}e^{2t}}$

$$\varphi y' + \varphi' y = (\varphi y)' = \underbrace{3\cancel{t^2}e^t\left(\frac{1}{\cancel{t^2}}\right)e^{2t}}$$

$$(\varphi y)' = 3e^{3t} \Rightarrow \varphi y = \int 3e^{3t}dt = e^{3t} + C$$

$$y = \frac{1}{\varphi}e^{3t} + C = \frac{e^{3t}+C}{\frac{1}{t^2}e^{2t}} = t^2e^t + Ct^2e^{-2t}$$

General sol-n:

$$y(t) = t^2e^t + Ct^2e^{-2t}.$$

From the initial condition $y(1) = e + (-2)e^{-2} \Rightarrow \boxed{C=-2}$

Solution to the initial value problem

$$\boxed{y(t) = t^2e^t - 2t^2e^{-2t}.}$$

10. *(10 points)* Find the general solution to

$$y'' + 3y' + 2y = 3e^{2t} + 1.$$

Homogeneous problem:

$$y'' + 3y' + 2y = 0$$

Char. polynomial: $r^2 + 3r + 2 = 0$

$$(r+1)(r+2) = 0$$

$y_1(t) = e^{-t}$, $y_2(t) = e^{-2t}$

Guess for the form of particular sol-n

$$Y(t) = Ae^{2t} + B$$

Note: there are no overlaps with y_1 and y_2.

Find A and B (plug $Y(t)$ into D.E.)

$$4A\cancel{e^{2t}} + 3(2A\cancel{e^{2t}}) + 2A\cancel{e^{2t}} = 3\cancel{e^{2t}}$$

$$A(4+6+2) = 3 \implies A = \frac{3}{12} = \frac{1}{4}$$

$$0 + 0 + 2B = 1 \implies B = \frac{1}{2}$$

$$Y(t) = \frac{1}{4}e^{2t} + \frac{1}{2}$$

General sol-n:

$$\boxed{y(t) = c_1 y_1(t) + c_2 y_2(t) + Y(t) = c_1 e^{-t} + c_2 e^{-2t} + \left(\frac{1}{4}e^{2t} + \frac{1}{2}\right)}$$

11. *(14 points)* Given that

$$f(t) = e^{3t}\cos t - 2te^{-t} + u_3(t)(t^2 - 1 + e^{2t}),$$

(a) Find the Laplace transform of $f(t)$.

$$\mathcal{L}\{e^{3t}\cos t\}(s) \overset{(14)}{=} \mathcal{L}\{\cos t\}(s-3) \overset{(6)}{=} \frac{(s-3)}{(s-3)^2+1^2}.$$

$$\mathcal{L}\{2te^{-t}\}(s) \overset{(14)}{=} 2\mathcal{L}\{t\}(s+1) \overset{(3)}{=} 2\frac{1}{(s+1)^2}.$$

$$\mathcal{L}\{u_3(t)(t^2-1+e^{2t})\}(s) \overset{(13)}{=} e^{-3s}\mathcal{L}\{(t+3)^2-1+e^{2(t+3)}\}(s) =$$

$$= e^{-3s}\mathcal{L}\{t^2+6t+8+e^6 e^{2t}\}(s) \overset{(3),\,(2)}{=}$$

$$= e^{-3s}\left[\frac{2!}{s^3} + 6\frac{1!}{s^2} + 8\frac{1}{s} + e^6\frac{1}{s-2}\right]$$

$$\boxed{\mathcal{L}\{f(t)\}(s) = \frac{(s-3)}{(s-3)^2+1} - 2\frac{1}{(s+1)^2} + e^{-3s}\left[\frac{2}{s^3} + \frac{6}{s^2} + \frac{8}{s} + \frac{e^6}{s-2}\right]}$$

(b) Find the values $f(\frac{\pi}{2})$ and $f(\pi)$.

Note: $\pi \approx 3.14\ldots$

$$f\left(\tfrac{\pi}{2}\right) = e^{\frac{3\pi}{2}}\cos\tfrac{\pi}{2} - 2\left(\tfrac{\pi}{2}\right)e^{-\pi/2} = \cancel{e^{\frac{3\pi}{2}} \cdot 0} - \pi e^{-\frac{\pi}{2}}$$

$$f(\pi) = e^{3\pi}\cos\pi - 2\pi e^{-\pi} + (\pi^2 - 1 + e^{2\pi}) =$$

$$= -e^{3\pi} - 2\pi e^{-\pi} + (\pi^2 - 1 + e^{2\pi})$$

12. *(20 points)* Find the inverse Laplace transform of

$$F(s) = \frac{(2s^2+1)e^{-s}}{(s-1)(s^2+4s+5)}.$$

Note $\quad s^2+4s+5 = (s^2+4s+4)+1 = (s+2)^2+1$

Partial fraction decomposition:

$$\frac{2s^2+1}{(s-1)(s^2+4s+5)} = \frac{A}{s-1} + \frac{Bs+C}{s^2+4s+5} = \frac{\frac{3}{10}}{s-1} + \frac{\frac{17}{10}s+\frac{1}{2}}{s^2+4s+5}$$

s^2: $\quad A+B=2 \quad \xrightarrow{+} \quad 5A+C=2$

s: $\quad 4A-B+C=0$

1: $\quad 5A-C=1$

$$\begin{cases} 10A=3 \\ 2C=1 \end{cases} \Rightarrow \boxed{A=\frac{3}{10}, \quad C=\frac{1}{2}}$$

$$\boxed{B = 2-A = \frac{17}{10}}$$

$$\mathcal{L}^{-1}\left\{e^{-s}\frac{2s^2+1}{(s-1)(s^2+4s+5)}\right\}(t) \overset{(13)}{=} u_1(t)\,\mathcal{L}^{-1}\left\{\frac{3/10}{s-1} + \frac{\frac{17}{10}s+\frac{1}{2}}{s^2+4s+5}\right\}(t-1) =$$

$$\overset{(2)}{=} u_1(t)\left[\frac{3}{10}e^{t-1} + \mathcal{L}^{-1}\left\{\frac{\frac{17}{10}(s+2)-\frac{34}{10}+\frac{1}{2}}{(s+2)^2+1^2}\right\}(t-1)\right] \overset{(14)}{=}$$

$$= u_1(t)\left[\frac{3}{10}e^{t-1} + e^{-2(t-1)}\mathcal{L}^{-1}\left\{\frac{\frac{17}{10}s-\frac{29}{10}}{s^2+1}\right\}(t-1)\right] \overset{(5),(6)}{=}$$

$$= \boxed{u_1(t)\left[\frac{3}{10}e^{t-1} + e^{2(t-1)}\left(\frac{17}{10}\cos(t-1) - \frac{29}{10}\sin(t-1)\right)\right]}$$

13. *(10 points)* A system with a mass of 2kg at the end of a spring with spring constant 11N/m is immersed in a medium with damping constant 6Ns/m. At $t = 0$, the mass is pulled down 10cm from the equilibrium position and released with an upward velocity 0.5m/s. An external force of 3N is applied to the system from $t = 1$s to $t = 4$s and then removed. At $t = 6$s an impulsive external force $\delta(t-6)$ is applied to the system.

$m = 2$

$\gamma = 6$

$k = 11$

(a) Give the differential equation with initial conditions for this motion using the $u_c(t)$ function.

$$m x'' + \gamma x' + k x = f$$

$$2x'' + 6x' + 11x = 3u_1(t) - 3u_4(t) + \delta(t-6)$$

$$f(t) = 3(u_1(t) - u_4(t)) + \delta(t-6)$$

Initial conditions $\begin{cases} x(0) = .1 \\ x'(0) = -.5 \end{cases}$

(b) Find the Laplace transform of the solution. DO NOT SOLVE FOR THE SOLUTION.

<u>L.H.S.</u> $\mathcal{L}\{2x'' + 6x' + 11x\}(s) = 2(s^2 X(s) - s x(0) - x'(0)) + 6(s X(s) - x(0)) + 11 X(s) =$

$= \underline{(2s^2 + 6s + 11) X(s) - .2s + .4}$

<u>R.H.S.</u> $\mathcal{L}\{3u_1(t) - 3u_4(t) + \delta(t-6)\} \overset{(13),(17)}{=}$

$= \underline{3\frac{e^{-s}}{s} - 3\frac{e^{-4s}}{s} + e^{-6s}}$

14. *(15 points)* Given the system

$$\begin{cases} x_1' = -2x_1 + x_2 \\ x_2' = x_1 - 2x_2 \end{cases} \qquad \begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

(a) Find the general solution of this system.

Eigenvalues: $(-2-\lambda)^2 - 1 = 0 \Rightarrow (2+\lambda)^2 = 1 \Rightarrow$

$$\Rightarrow \lambda_{1,2} = -2 \pm 1 = \left[\begin{matrix} -1 \\ -3 \end{matrix}\right.$$

Eigenvectors:

$\lambda_1 = -1$

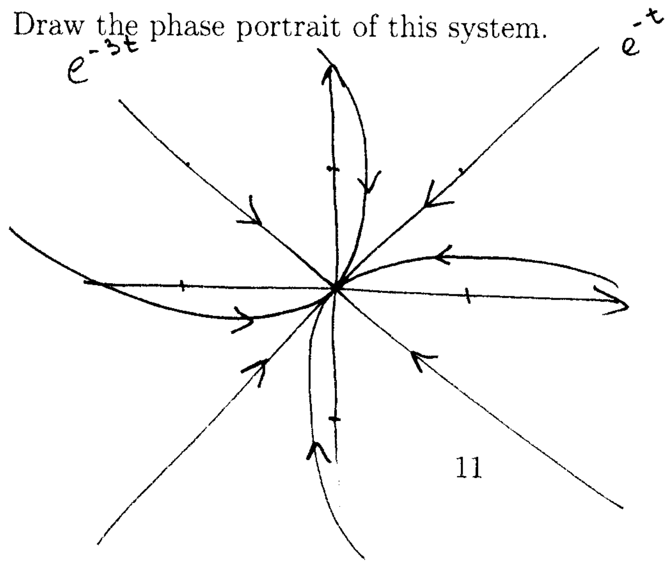
$$\left[\begin{array}{cc|c} -1 & 1 & 0 \\ 0 & \underset{\substack{\uparrow \\ \text{free}}}{0} & 0 \end{array}\right] \qquad \begin{bmatrix} +x_2 \\ x_2 \end{bmatrix} = \begin{bmatrix} +1 \\ 1 \end{bmatrix} x_2$$

$\lambda_2 = -3$

$$\left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & \underset{\substack{\uparrow \\ \text{free}}}{0} & 0 \end{array}\right] \qquad \begin{bmatrix} -x_2 \\ x_2 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} x_2$$

General sol-n: $\boxed{\vec{x}(t) = c_1 e^{-t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{-3t} \begin{bmatrix} -1 \\ 1 \end{bmatrix}}$

(b) Draw the phase portrait of this system.



$$\begin{bmatrix} 3+i \\ 2i \end{bmatrix} = \underbrace{\begin{bmatrix} 3 \\ 0 \end{bmatrix}}_{\vec{a}} + i \underbrace{\begin{bmatrix} 1 \\ 2 \end{bmatrix}}_{\vec{b}}$$

15. *(10 points)* Suppose A is a 2×2 matrix with real entries such that

$$A \begin{bmatrix} 3+i \\ 2i \end{bmatrix} = (2+i) \begin{bmatrix} 3+i \\ 2i \end{bmatrix}.$$

(a) Find the general solution of the system $\mathbf{x}' = A\mathbf{x}$.

Two basic solutions

$$\vec{x}_1(t) = e^{2t}\left(\vec{a}\cos t - \vec{b}\sin t\right) =$$

$$= e^{2t}\left(\begin{bmatrix} 3 \\ 0 \end{bmatrix}\cos t - \begin{bmatrix} 1 \\ 2 \end{bmatrix}\sin t\right),$$

$$\vec{x}_2(t) = e^{2t}\left(\vec{a}\sin t + \vec{b}\cos t\right) =$$

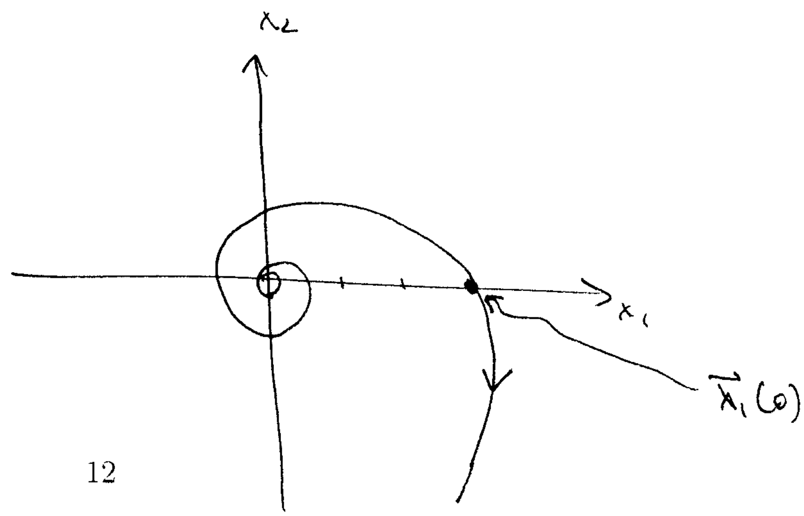
$$= e^{2t}\left(\begin{bmatrix} 3 \\ 0 \end{bmatrix}\sin t + \begin{bmatrix} 1 \\ 2 \end{bmatrix}\cos t\right),$$

General solution

$$\boxed{\vec{x}(t) = c_1\vec{x}_1(t) + c_2\vec{x}_2(t)}$$

(b) Draw a possible phase portrait of this system.

Draw $\vec{x}_1(t)$

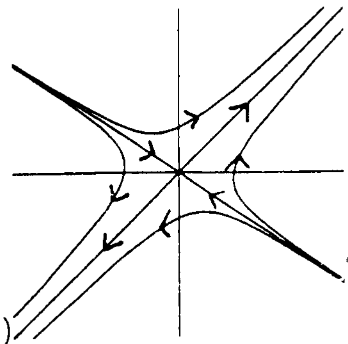


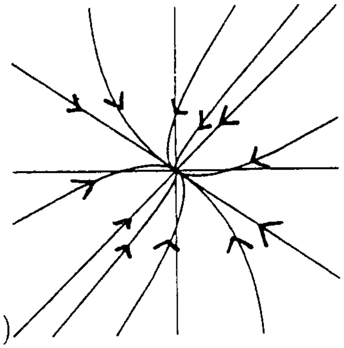
16 *(21 points)* Classify the type of the origin, determine the stability of the system and choose a phase portrait (given on the next page) of the system $\mathbf{x}' = \mathbf{Ax}$, where the matrix $\mathbf{A}$ has the following eigenvalues r_1, r_2.

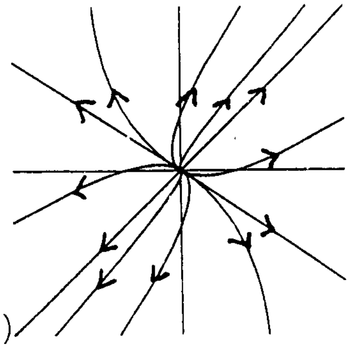
	EIGENVALUES	TYPE	STABILITY	PHASE PORTRAIT
(a)	$r_1 = -1, r_2 = 2$	Saddle point	unstable	A
(b)	$r_1 = 2, r_2 = 3$	nodal source	unstable	C
(c)	$r_1 = r_2 = -3$ with two linearly independent eigenvectors	proper node	asympt. stable	G
(d)	$r_1 = r_2 = -5$ with only one linearly independent eigenvector	improper node	asympt. stable	D
(e)	$r_1 = -2, r_2 = -5$	nodal sink	asympt. stable	B
(f)	$r_1 = 3i, r_2 = -3i$	center	stable	F
(g)	$r_1 = -2 + i, r_2 = -2 - i$	spiral sink	asympt. stable	J

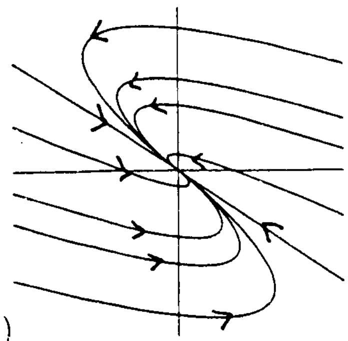
The possible types are: saddle point, node, center, spiral point, proper node, and improper node.

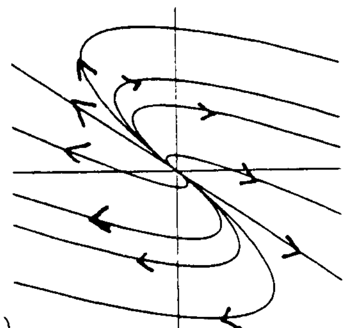
The system may be: stable, asymptotically stable, or unstable.

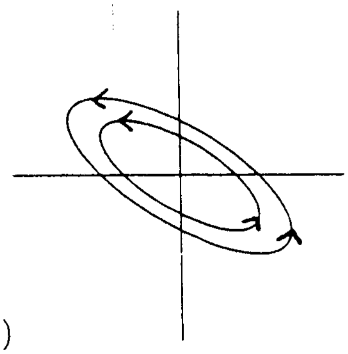

(A)

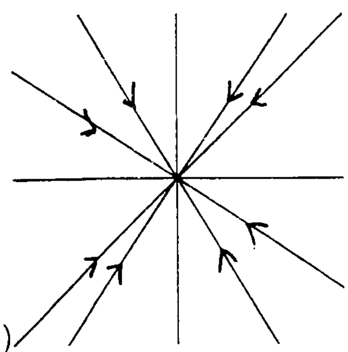

(B)

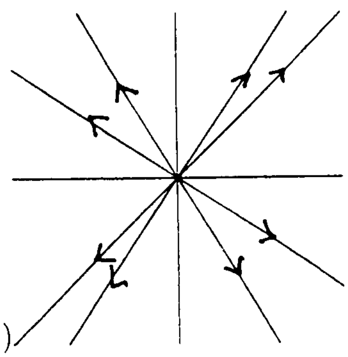

(C)

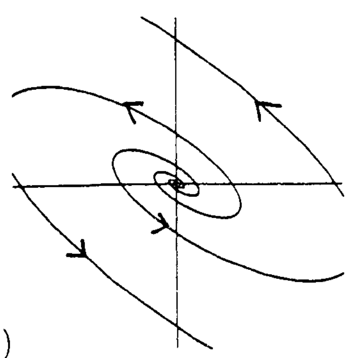

(D)

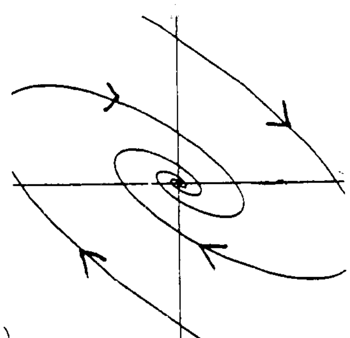

(E)


(F)


(G)


(H)


(I)


(J)

$f(t) = \mathcal{L}^{-1}\{F(s)\}$	$F(s) = \mathcal{L}\{f(t)\}$
1. 1	$\dfrac{1}{s}, \quad s > 0$
2. e^{at}	$\dfrac{1}{s-a}, \quad s > a$
3. t^n; $\quad n =$ positive integer	$\dfrac{n!}{s^{n+1}}, \quad s > 0$
4. t^p, $p > -1$	$\dfrac{\Gamma(p+1)}{s^{p+1}}, \quad s > 0$
5. $\sin at$	$\dfrac{a}{s^2+a^2}, \quad s > 0$
6. $\cos at$	$\dfrac{s}{s^2+a^2}, \quad s > 0$
7. $\sinh at$	$\dfrac{a}{s^2-a^2}, \quad s > \|a\|$
8. $\cosh at$	$\dfrac{s}{s^2-a^2}, \quad s > \|a\|$
9. $e^{at}\sin bt$	$\dfrac{b}{(s-a)^2+b^2}, \quad s > a$
10. $e^{at}\cos bt$	$\dfrac{s-a}{(s-a)^2+b^2}, \quad s > a$
11. $t^n e^{at}$, $n =$ positive integer	$\dfrac{n!}{(s-a)^{n+1}}, \quad s > a$
12. $u_c(t)$	$\dfrac{e^{-cs}}{s}, \quad s > 0$
13. $u_c(t)f(t-c)$	$e^{-cs}F(s)$
14. $e^{ct}f(t)$	$F(s-c)$
15. $f(ct)$	$\dfrac{1}{c}F\left(\dfrac{s}{c}\right), \quad c > 0$
16. $\int_0^t f(t-\tau)g(\tau)\,d\tau$	$F(s)G(s)$
17. $\delta(t-c)$	e^{-cs}
18. $f^{(n)}(t)$	$s^nF(s) - s^{n-1}f(0) - \cdots - f^{(n-1)}(0)$
19. $(-t)^n f(t)$	$F^{(n)}(s)$