Solutions

Name:

MATH 250

Student Number:

Midterm Exam II

Instructor:

November 07, 2005

Section:

This exam has 12 questions for a total of 100 points. There are **4** partial credit questions. **In order to obtain full credit for partial credit problems, all work must be shown. Credit will not be given for an answer not supported by work.**

THE USE OF CALCULATORS or ANY OTHER ELECTRONIC DEVICES IS NOT PERMITTED IN THIS EXAMINATION.
At the end of the examination, the booklet will be collected.

Do not write in this box.

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1. (5 points) Which of the following pairs of functions is linearly dependent?

   (a) $f(t) = \cos 2t$, $\quad g(t) = \sin 2t$;

   (b) $f(t) = e^{2t}$, $\quad g(t) = e^{-2t}$;

   (c) $f(t) = \cos t$, $\quad g(t) = \sin\left(t - \frac{\pi}{2}\right)$ $= -\cos(t)$

   (d) $f(t) = t^2$, $\quad g(t) = 2t$.

2. (5 points) Suppose y_1 and y_2 are two solutions of the second order linear differential equation

$$t\,y'' + 3\,y' + (2\sin t)\,y = 0.$$

What is their Wronskian $W(y_1, y_2)(t)$ as a function of time if $W(y_1, y_2)(1) = 2$?

   (a) $W(y_1, y_2)(t) = 2t^3$;

   (b) $W(y_1, y_2)(t) = 2e^{-t^3}$;

   (c) $W(y_1, y_2)(t) = \frac{2}{t^3}$;

   (d) $W(y_1, y_2)(t) = 2e^{t^3-1}$.

$$y'' + \underbrace{\frac{3}{t}}_{p(t)} y' + \frac{2\sin t}{t} y = 0$$

$$W(y_1, y_2) = C \cdot e^{-\int p(t)\,dt} = C \cdot e^{-\int \frac{3}{t}\,dt} = $$

$$= C \cdot e^{-3\ln t} =$$

$$= C\, t^{-3} = \boxed{\frac{C}{t^3}}$$

3. (5 points) Consider the following initial value problem describing a mechanical system:

$$2y'' + 8y = 0, \quad y(0) = 0 \text{ m}, \quad y'(0) = 6 \text{ m/s}.$$

What is the amplitude of the resulting oscillation?

(a) 2 meters;

(b) 3 meters;

(c) 4 meters;

(d) 6 meters.

(Undamped free vibration)

$y'' + 4y = 0, \quad r_{1,2} = \pm 2i$

$y = A\cos(2t) + B\sin(2t)$

From initial conditions

$y(0) = \boxed{0 = A}$

$y'(0) = 6 = 2B \implies \boxed{B = 3}$

$R = \sqrt{A^2 + B^2} = \sqrt{0 + 3^2} = 3$

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4. (5 points) Let $y(t)$ be a solution to the differential equation

$$y'' + 4y' + 4y = 0.$$

Which one of the following statements concerning $\lim_{t\to\infty} y(t)$ is **true**?

(a) Sometimes $y(t) \to 0$ and sometimes $y(t) \to \infty$ depending on the initial conditions $y(t)$ satisfies.

(b) For any solution $y(t)$, the limit is always zero.

(c) The limit does not exist because $y(t)$ oscillates.

(d) The limit is never zero for any solution $y(t)$.

Either remember the homework problem or note that the equation describes free oscillations with damping (friction)

5. (5 points) Which one of the following describes the general solution to the differential equation

$$9y'' - 12y' + 4y = 2e^{\frac{2}{3}t} ?$$

C_1, C_2 and A below are constants.

(a) $y = C_1e^{\frac{2}{3}t} + C_2te^{\frac{2}{3}t} + Ae^{\frac{2}{3}t}$;

(b) $y = C_1e^{\frac{2}{3}t} + C_2te^{\frac{2}{3}t} + Ate^{\frac{2}{3}t}$;

(c) $y = C_1e^{\frac{2}{3}t} + C_2te^{\frac{2}{3}t} + At^2e^{\frac{2}{3}t}$;

(d) $y = C_1e^{\frac{2}{3}t} + C_2e^{\frac{2}{3}t} + Ate^{\frac{2}{3}t}$.

Homogeneous equation

$$9y'' - 12y' + 4y = 0$$

Characteristic equation

$$9r^2 - 12r + 4 = 0$$

$$\boxed{r_{1,2}} = \frac{+12 \pm \sqrt{(12)^2 - 4\cdot 9\cdot 4}}{2\cdot 9} =$$

$$= \frac{12}{2\cdot 9} = \boxed{\frac{2}{3}}$$

$$y_1 = e^{\frac{2}{3}t}$$

$$y_2 = te^{\frac{2}{3}t}$$

Particular solution

$$Y = At^2e^{\frac{2}{3}t}$$

$$\omega = \frac{1}{3}$$

6. (5 points) Consider the system of forced vibrations $27y'' + ky = 4\cos(t/3)$. For what value of k will the system undergo resonance?

(a) 3;

(b) $3\sqrt{3}$;

(c) 4;

(d) 9.

Natural frequency of the system is determined from L.H.S.

$$27y'' + ky = 0$$

$$y = A\cos\left(\sqrt{\frac{k}{27}}\,t\right) + B\sin\left(\underbrace{\sqrt{\frac{k}{27}}}_{=\omega_0}\,t\right)$$

Resonance occurs when $\boxed{\omega = \omega_0}$

$$\sqrt{\frac{k}{27}} = \frac{1}{3} \Rightarrow \frac{k}{27} = \frac{1}{9} \Rightarrow \boxed{k = 3}$$

7. (5 points) Find a suitable form of a particular solution of

$$y'' + y = 2e^{-t} + \cos t + 2t.$$

A_1, A_2, A_3, A_4, and A_5 below are constants.

(a) $Y(t) = A_1e^{-t} + A_2\cos t + A_3t$,

(b) $Y(t) = A_1e^{-t} + A_2\sin t + A_3\cos t + A_4t + A_5$,

(c) $Y(t) = A_1e^{-t} + A_2t\sin t + A_3t\cos t + A_4t + A_5$,

(d) $Y(t) = A_1\cancel{te^{-t}} + A_2t\sin t + A_3t\cos t + A_4t$.

Homogeneous eq-n:

$y'' + y = 0$

$r^2 + 1 = 0$

$r_{12} = \pm i$

$y_1 = \cos t$

$y_2 = \sin t$

Particular solutions:

- $2e^{-t} \Rightarrow Y_1 = A_1e^{-t}$
- $\cos t \Rightarrow Y_2 = A_2t\cos t + A_3t\sin t$
- $2t \Rightarrow Y_3 = A_4t + A_5$

8. (5 points) Consider the differential equation

$$4y'' + 4y' + cy = 0.$$

Find the value of c so that the function

$$y(t) = C_1e^{-\frac{t}{2}} + C_2te^{-\frac{t}{2}}, \qquad (*)$$

where C_1 and C_2 are constants, is a solution of this equation.

(a) $c = 2$;

(b) $c = 1$;

(c) $c = \frac{1}{2}$;

(d) $c = -\frac{1}{4}$.

Characteristic equation

$4r^2 + 4r + c = 0 \Rightarrow r_{12} = \dfrac{-4 \pm \sqrt{16 - 4\cdot 4c}}{2\cdot 4} =$

$r_{1,2} = \left(-\frac{1}{2}\right) \pm \frac{1}{2}\sqrt{1-c}$

Solution $(*)$ corresponds to repeated roots $\left(-\frac{1}{2}\right) \Rightarrow$

$\boxed{c = 1}$

9. (15 points) Consider the following differential equation

$$ty'' + (1-2t)y' + (t-1)y = 0, \quad t > 0.$$

(a) (2 points) Verify that $y_1(t) = e^t$ is a solution to the given equation.

$$te^t + (1-2t)e^t + (t-1)e^t = e^t[t+1-2t+t-1] = 0.$$

(b) (10 points) Find a second linearly independent solution $y_2(t)$ to the given equation. **Explain** why the given function $y_1(t)$ and obtained function $y_2(t)$ are **linearly independent**.

Standard form: $y'' + (\frac{1}{t} - 2)y' + (1 - \frac{1}{t})y = 0$

Look for a solution in the form

$y_2 = v y_1$

Using <u>Abel's Thm</u>:

$$W(y_1, y_2) = W(y_1, vy_1) = \det \begin{vmatrix} y_1 & vy_1 \\ y_1' & vy_1' + v'y_1 \end{vmatrix} = y_1^2 v'$$

$$\| $$

$$Ce^{-\int p(t)\,dt}$$

$$\| $$

$$Ce^{-\int (\frac{1}{t} - 2)\,dt}$$

$$\| $$

$$Ce^{-\ln|t| + 2t}$$

$$= ct^{-1}e^{2t}$$

$$v' = \frac{1}{y_1^2} t^{-1} e^{2t} = \frac{1}{\cancel{e^{2t}}} t^{-1} \cancel{e^{2t}} = t^{-1}$$

$$v = \int t^{-1}\,dt = \ln t$$

$$\boxed{y_2 = vy_1 = (\ln t)e^t}$$

(c) (3 points) What is the general solution of the given equation?

$$\boxed{y(t) = Ay_1(t) + By_2(t) = Ae^t + B(\ln t)e^t}$$

10. (15 points) Use the method of undetermined coefficients to find the general solution of the following differential equation

$$y'' + y' - 2y = 2te^t.$$

Homogeneous equation: $y'' + y' - 2y = 0$

Char. eq.: $r^2 + r - 2 = 0$, $r_{1,2} = \frac{-1 \pm \sqrt{1^2 + 4 \cdot 2}}{2} =$

$= \frac{-1 \pm \sqrt{9}}{2} = \frac{-1 \pm 3}{2} = \begin{cases} 1 \\ -2 \end{cases}$

$\boxed{y_1 = e^t \quad y_2 = e^{-2t}}$

Initial guess for a particular solution:

$Y = (At + B)e^t$

After noticing repetition in $y_1 = e^t$, corrected guess

$Y = t(At + B)e^t = (At^2 + Bt)e^t$

Substituting:

$[2A + 2(2At + B) + (At^2 + Bt)]e^t +$

$+ [(2At + B) + (At^2 + Bt)]e^t - 2(At^2 + Bt)e^t = 2te^t$

Divide through by e^t.

Group the terms with t^2, t, and 1:

$$\left.\begin{aligned} & t^2\cdot\left[A+A-2A\right]+ \\ & +t\cdot\left[4A+B+2A+B-2B-2\right]+ \\ & +1\cdot\left[2A+2B+B\right] \end{aligned}\right\} = 0$$

Since t^2, t, and 1 are linearly independent functions each coefficient must be 0. $\left[A+A-2A\right]=0$ automatically

$$\begin{cases} 6A-2=0 \implies A=\frac{1}{3} \\ 2A+3B=0 \implies B=-\frac{2}{3}A=-\frac{2}{9} \end{cases}$$

Particular solution

$$Y=\left(\frac{1}{3}t^2-\frac{2}{9}t\right)e^t$$

General solution:

$$\boxed{y(t)=C_1y_1+C_2y_2+Y=}$$

$$\boxed{=C_1e^t+C_2e^{-2t}+\left(\frac{1}{3}t^2-\frac{2}{9}t\right)e^t}$$

11. (20 points) A mass of $m = 2\ kg$ is suspended by a spring with spring constant $k\ N/m$. The system is immersed in a fluid with damping constant $\gamma\ kg/sec$. In absence of external force, the differential equation for this system is

$$mu'' + \gamma u' + ku = 0,$$

where $u(t)$ is the displacement of the mass from the equilibrium position at time t. Assume that g, the acceleration due to gravity, is $10\ m/sec^2$.

(a) (3 points) Suppose the mass of 2 kg stretches the spring 40 cm. What is k?

$$K = \frac{mg}{L} = \frac{2\cdot 10}{.4} = \boxed{50}$$

(b) (3 points) For what value(s) of γ will the system be critically damped?

$$\gamma^2 - 4mk = 0, \quad \gamma_{critical} = \sqrt{4mk} = \sqrt{4\cdot 2\cdot 50} = \boxed{20}$$

(c) (10 points) Determine the position of the mass if the mass is released from the equilibrium position with an upward velocity of 1 m/sec and the damping constant $\gamma = 16\ kg/sec$.

$u_0 = 0$, (from equilibrium position)

$u_0' = -1$ (upward)

$$mr^2 + \gamma r + K = 0 \Rightarrow r_{1,2} = \frac{-\gamma}{2m} \pm \frac{\sqrt{\gamma^2 - 4mk}}{2m} =$$

$$= \frac{-16}{2\cdot 2} \pm \frac{\sqrt{16^2 - 4\cdot 2\cdot 50}}{2\cdot 2} = -4 \pm \sqrt{64 - 10} = -4 \pm \sqrt{-36} =$$

$$= -4 \pm 6i$$

$$u(t) = e^{-4t}\left[A\cos(6t) + B\sin(6t)\right]$$

$$u(0) = \boxed{0 = A}$$

$$u'(0) = -4A + 6B = -1 \Rightarrow B = \frac{-1}{6}$$

$$u(t) = \frac{-1}{6}e^{-4t}\sin(6t) =$$

$$= \frac{1}{6}e^{-4t}\cos(6t + \pi/2)$$

(d) (4 points) Find the quasi-period and quasi-frequency of this motion.

Quasi-frequency: 6

Quasi-period: $\frac{2\pi}{6} = \frac{\pi}{3}$

12. (10 points) Use the definition to find the Laplace transform of the function

$$f(t) = t - 2.$$

$$F(s) = \mathcal{L}\{f(t)\}(s) = \int_0^\infty f(t)\, e^{-ts}\, dt =$$

$$= \int_0^\infty (t-2)e^{-ts}\, dt = \left[\text{Integration by parts}\right] =$$

$$= \int_0^\infty (t-2)\frac{d}{dt}\left(\frac{-1}{s}e^{-ts}\right) dt = (t-2)\left(\frac{-1}{s}e^{-ts}\right)\Big|_{t=0}^{\infty} -$$

$$- \int_0^\infty \left(\frac{-1}{s}\right) e^{-ts}\, dt = -(0-2)\left(\frac{-1}{s}e^{-0\cdot s}\right) -$$

$$- \left(\frac{-1}{s}\right)^2 e^{-ts}\Big|_{t=0}^{\infty} = -\frac{2}{s} + \left(\frac{-1}{s}\right)^2 = \boxed{\frac{1-2s}{s^2}}$$