10 pts 1. Determine the equation of the plane containing the two parallel lines

$$L_1: \quad x = -6t, \quad y = 1 + 9t, \quad z = -3t$$

and

$$L_2: \quad x = 1 + 2s, \quad y = 4 - 3s, \quad z = s.$$

Write the equation in standard form: $ax + by + cz + d = 0$.

Find the direction vectors of the lines

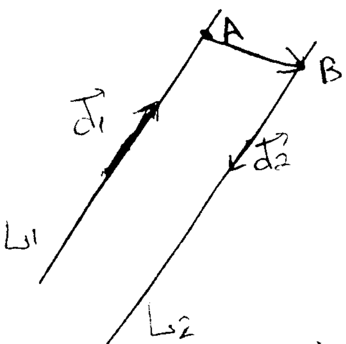
$L_1: \vec{r} = \langle 0, 1, 0 \rangle + t\langle -6, 9, -3 \rangle$

$L_2: \vec{r} = \langle 1, 4, 0 \rangle + s\langle 2, -3, 1 \rangle$

the direction vectors $\vec{d_1} = \langle -6, 9, -3 \rangle$

$\vec{d_2} = \langle 2, -3, 1 \rangle$

($\vec{d_1} = -3\vec{d_2}$, so the lines are indeed parallel).



↑ just a sketch, not in any coordinate system.

Find any point on the first line. Say, $t = 0 \Rightarrow A = (-6 \cdot 0, 1 + 9 \cdot 0, -3 \cdot 0) =$
$= (0, 1, 0) \in L_1$ - lies on L_1.

Similarly,
$B = (1, 4, 0) \in L_2$.

Vector for $\vec{AB} = \langle 1, 4, 0 \rangle - \langle 0, 1, 0 \rangle = \langle 1, 3, 0 \rangle$

$\vec{AB}$ is parallel to the plane of L_1, L_2, and so are both $\vec{d_1}$, $\vec{d_2}$.

$\Rightarrow$ for instance, $\vec{d_2} \times \vec{AB}$ is orthogonal to that plane. Let $\vec{n} = \vec{d_2} \times \vec{AB} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & -3 & 1 \\ 1 & 3 & 0 \end{vmatrix} =$

$$= \begin{vmatrix} -3 & 1 \\ 3 & 0 \end{vmatrix} \vec{i} - \begin{vmatrix} 2 & 1 \\ 1 & 0 \end{vmatrix} \vec{j} + \begin{vmatrix} 2 & 1 \\ -3 & +3 \end{vmatrix} \vec{k} =$$

$$= -3\vec{i} + \vec{j} - 3\vec{k} = \langle -3, 1, +9 \rangle$$

Check for certainty:

$\vec{n} \cdot \vec{d_2} = \langle -3, 1, +9 \rangle \cdot \langle 2, -3, 1 \rangle = (-3)(2) +$
$(1)(-3) + (+9)(1) = 0$

$\vec{n} \cdot \overrightarrow{AB} = \langle -3, 1, 9 \rangle \cdot \langle 1, 3, 0 \rangle = (-3)(1) + (1)(3) +$
$(9)(0) = 0.$

Finally, pick any point in the plane, say, $A(0, 1, 0)$

$\vec{r_0} = \langle 0, 1, 0 \rangle$, so $\vec{n} \cdot (\vec{r} - \vec{r_0}) = 0$, i.e.

$-3(x - 0) + 1(y - 1) + 9(z - 0) = 0$, i.e.

$$\underline{-3x + y + 9z - 1 = 0}$$

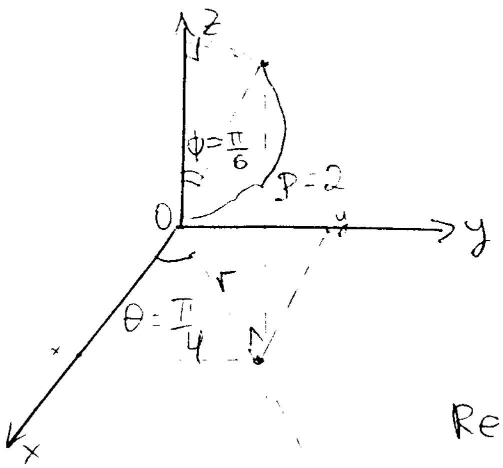
For certainty check $B(1, 4, 0)$:

$-3(1) + (4) + 9(0) - 1 = 0$, indeed.

10 pts 2.

6 pts a) Convert the given set of spherical coordinates to cylindrical coordinates and to rectangular coordinates.

$$(\rho, \theta, \phi) = (2, \frac{\pi}{4}, \frac{\pi}{6})$$



Spherical → Rectangular

$z = \rho \cos\phi = 2 \cdot \frac{\sqrt{3}}{2} = \sqrt{3}$

$r = \rho \sin\phi = 2 \cdot \frac{1}{2} = 1$

$x = r\cos\theta = 1 \cdot \frac{\sqrt{2}}{2} = \frac{\sqrt{2}}{2}$

$y = r\sin\theta = 1 \cdot \frac{\sqrt{2}}{2} = \frac{\sqrt{2}}{2}$

Rectangular: (x, y, z) $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, \sqrt{3}\right)$

Cylindrical: (r, θ, z) $\left(1, \frac{\pi}{4}, \sqrt{3}\right)$

4 pts b) Write the given equation in rectangular coordinates:

$$\rho^2(2\sin^2\phi - \cos^2\phi) = 1$$

$z = \rho\cos\phi \Rightarrow z^2 = \rho^2\cos^2\phi$

$r = \rho\sin\phi \Rightarrow r^2 = \rho^2\sin^2\phi$

$r^2 = x^2 + y^2$, so the equation

$\rho^2(2\sin^2\phi - \cos^2\phi) = 1$ now looks like

$2\rho^2\sin^2\phi - \rho^2\cos^2\phi = 1$, or

$2(x^2 + y^2) - z^2 = 1$

12 pts 3. Consider the curve

$$\mathbf{r}(t) = e^t\mathbf{i} + e^{-t}\mathbf{j} + \sqrt{2}t\mathbf{k} \qquad 0 \leqslant t \leqslant 2$$

8 pts a) Find the unit tangent and unit normal vectors of this curve, both as functions of time.

$$\vec{r}\,'(t) = e^t\vec{i} - e^{-t}\vec{j} + \sqrt{2}\,\vec{k}$$

$$|\vec{r}\,'(t)| = \sqrt{e^{2t} + e^{-2t} + 2} = \sqrt{(e^t+e^{-t})^2} = e^t + e^{-t}$$

$$\vec{T}(t) = \frac{\langle e^t, -e^{-t}, \sqrt{2}\rangle}{\sqrt{e^{2t}+e^{-2t}+2}} = \frac{\langle e^t, -e^{-t}, \sqrt{2}\rangle}{\sqrt{(e^t+e^{-t})^2}} =$$

$$= \underline{\frac{1}{e^t+e^{-t}} \cdot \langle e^t, -e^{-t}, \sqrt{2}\rangle}$$ – the Unit tangent Vector.

$$\vec{T}\,'(t) = \frac{d}{dt}\left\langle \frac{e^t}{e^t+e^{-t}}, \frac{-e^{-t}}{e^t+e^{-t}}, \frac{\sqrt{2}}{e^t+e^{-t}} \right\rangle =$$

$$= \left\langle \frac{e^t(e^t+e^{-t}) - (e^t - e^{-t})e^t}{(e^t+e^{-t})^2}, \frac{e^{-t}(e^t+e^{-t}) - (e^t-e^{-t})(-e^{-t})}{(e^t+e^{-t})^2}, \ldots \right.$$

4 pts b) Find the curvature of this curve as a function of time.

See next page

$$\ldots \frac{-\sqrt{2}(e^{t}-e^{-t})}{(e^{t}+e^{-t})^{2}} \rangle$$

So $\vec{T}'(t) = \frac{1}{(e^{t}+e^{-t})^{2}} \langle 2, 2, -\sqrt{2}(e^{t}-e^{-t}) \rangle$

$$|\vec{T}'(t)| = \frac{1}{(e^{t}+e^{-t})^{2}} \sqrt{4+4+2(e^{t}-e^{-t})^{2}}$$

Hence, $\vec{N}(t) = \frac{\vec{T}'(t)}{|\vec{T}'(t)|} = \frac{1}{\sqrt{8+2(e^{t}-e^{-t})^{2}}} \cdot \langle 2, 2, -\sqrt{2}(e^{t}-e^{-t}) \rangle$

$$\kappa(t) = \frac{|\vec{T}'(t)|}{|\vec{r}'(t)|} = \frac{1}{(e^{t}+e^{-t})^{2}} \frac{\sqrt{8+2(e^{t}-e^{-t})^{2}}}{e^{t}+e^{-t}} =$$

$$= \frac{\sqrt{8+2(e^{t}-e^{-t})^{2}}}{(e^{t}+e^{-t})^{3}}$$

8 pts 4. Find the arclength of the following curve for $1 \leqslant t \leqslant 3$.

$$\mathbf{r}(t) = \frac{t^2}{2}\mathbf{i} + \frac{2\sqrt{2}}{3}t^{3/2}\mathbf{j} + t\mathbf{k}$$

$$\vec{r}\,'(t) = \langle t, \sqrt{2t}, 1 \rangle$$

$$|\vec{r}\,'(t)| = \sqrt{t^2 + 2t + 1} = \sqrt{(t+1)^2} = t+1,$$

as far as

$1 \le t \le 3 \Rightarrow t+1 \geq 0$.

So $$\int_1^3 |\vec{r}\,'(t)|\,dt = \int_1^3 (t+1)\,dt = \left(\frac{t^2}{2} + t\right)\Big|_1^3 =$$

$$= \left(\frac{3^2}{2} + 3\right) - \left(\frac{1^2}{2} + 1\right) = 7.5 - 1.5 = \boxed{6}$$

10 pts 5. Consider a particle whose acceleration is given by

$$\mathbf{a}(t) = -\cos t\mathbf{i} - \sin t\mathbf{j} + (2t-1)\mathbf{k}$$

With initial velocity $\mathbf{v}(0) = \langle 0,1,2\rangle$ and initial position $\mathbf{r}(0) = \langle 1,0,0\rangle$.

4 pts a) Find the velocity of the particle as a function of time.

$$\vec{v}(t) = \vec{v}(0) + \int_0^t \vec{a}(u)\,du = \langle 0,1,2\rangle +$$

$$\left(\langle -\sin(u), \cos(u), u^2-u\rangle \Big|_0^t \right) =$$

$$= \langle 0,1,2\rangle + \langle -\sin(t), \cos(t), t^2-t\rangle -$$

$$- \langle \underset{-\sin(0)}{0}, \underset{\cos(0)}{1}, \underset{0^2-0}{0}\rangle = \underline{\langle -\sin(t), \cos(t), t^2-t+2\rangle} = \vec{v}(t)$$

Check: $\vec{v}\,'(t) = \vec{a}(t)$, $\vec{v}(0) = \langle 0,1,2\rangle$ indeed.

4 pts b) Find the position of the particle as a function of time.

$$\vec{r}(t) = \vec{r}(0) + \int_0^t \vec{v}(u)\,du = \langle 1,0,0\rangle +$$

$$+ \left(\langle \cos(u), \sin(u), \frac{u^3}{3} - \frac{u^2}{2} + 2u\rangle \Big|_0^t \right) =$$

$$= \langle 1,0,0\rangle + \langle \cos(t), \sin(t), \frac{t^3}{3} - \frac{t^2}{2} + 2t\rangle -$$

$$\langle \underset{\cos(0)}{1}, \underset{\sin(0)}{0}, \underset{\frac{0^3}{3}-\frac{0^2}{2}+2\cdot 0}{0}\rangle = \underline{\langle \cos(t), \sin(t), t^3/3 - t^2/2 + 2t\rangle} = \vec{r}(t)$$

Check: $\vec{r}\,'(t) = \vec{v}(t)$, $\vec{r}(0) = \langle 1,0,0\rangle$ indeed.

2 pts c) Find the speed at time $t = 1$.

$$|\vec{v}(1)| = |\langle -\sin(1), \cos(1), 1^2 - 1 + 2\rangle| =$$

$$\sqrt{\sin^2(1) + \cos^2(1) + 4} = \boxed{\sqrt{5}}$$

10 pts 6. Determine whether the following limits exist. Prove your claim.

We haven't covered this material yet.

5 pts a)

$$\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x+y},$$

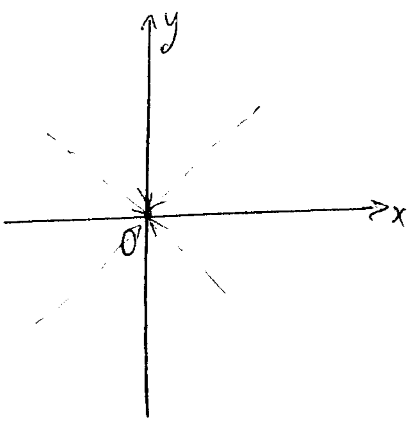
$$\frac{x^2-y^2}{x+y} = \frac{(x-y)(x+y)}{x+y} = x-y \underset{(x,y)\to(0,0)}{\longrightarrow} 0-0=0$$

this is true for any (x,y), as far as $x \neq y$

5 pts b)

$$\lim_{(x,y)\to(0,0)} \frac{3x^3y}{x^4+2y^4}.$$

approach the origin along $y=x$



then the lim $\to \lim\limits_{x\to 0} \frac{3x^4}{x^4+2x^4} = 1$

if along $y=-x$,

then lim $= \lim \frac{-3x^4}{x^4+2x^4} = -1$

$\Rightarrow$ the limit does not exist.

10 pts 7. Find all the second partial derivatives of

$$f(x, y) = e^{-xy} \cos x.$$

We haven't covered this material yet.

10 pts 8. Find the parametric equations of the line tangent to the curve of intersection of the two surfaces, $z = x^2 + 3y^2$ and $x + y + z = 4$ at the point $(1, -1, 4)$.

$\begin{cases} z = x^2 + 3y^2 \\ x + y + z = 4 \end{cases}$ Plug in z $\begin{cases} z = x^2 + 3y^2 \\ x + y + x^2 + 3y^2 = 4 \end{cases}$

Cylinder

Complete squares

$$\left(x^2 + x + \frac{1}{4}\right) + 3y^2 + y + \frac{1}{4 \cdot 3} = \underbrace{4 - \frac{1}{4} - \frac{1}{12}}_{= \frac{11}{3}}$$

$$(x^{2} + \tfrac{1}{2})^2 + 3(y + \ )^2 = \frac{11}{3}$$

$$x(t) = -\frac{1}{2} + \sqrt{\frac{11}{3}} \cos t$$

$$y(t) = -\frac{1}{6} + \frac{\sqrt{11}}{3} \sin t \qquad z(t) = x^2(t) + 3y^2(t) = 4 - x(t) - y(t) =$$

$$= \frac{14}{3} - \sqrt{\frac{11}{3}} \cos t - \frac{\sqrt{11}}{3} \sin t$$

$$\vec{r}'(t) = \begin{bmatrix} -\sqrt{\frac{11}{3}} \sin t \\ + \frac{\sqrt{11}}{3} \cos t \\ \sqrt{\frac{11}{3}} \sin t - \frac{\sqrt{11}}{3} \cos t \end{bmatrix}$$

Find t_0 s.t. $\vec{r}(t_0) = \begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix}$.

Compute the eq-n of the line.

10 pts 9. For each problem circle T if the statement is true and F if the statement is false.

a) T F A direction and a point always determine a unique line.

b) T F Three distinct points always determine a unique plane.

c) T F Two planes perpendicular to a third plane are parallel to each other.

d) T F Hyperbolas have negative curvature.

e) T F The vectors $\langle t, t\sin^2 t, -\cos t\rangle$ and $\langle -1/t, 1/t, -\cos t\rangle$ are perpendicular for $t \neq 0$.

f) T F $(2-x)^2 + (-3-y)^2 + (5-z)^2 = 6$ is a sphere of radius $\sqrt{6}$ and center $(-2, 3, -5)$.

g) T F $\mathbf{r}(t) \times |\mathbf{r}'(t) \times \mathbf{r}''(t)|$ is a vector.

h) T F For all vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$: $|\mathbf{c} \cdot (\mathbf{b} \times \mathbf{a})| = |\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})|$.

i) T F If θ is the angle between the vectors $\mathbf{a}$ and $\mathbf{b}$ then $\cos\theta = \dfrac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}||\mathbf{b}|}$.

j) T F f_{xy} and $\dfrac{\partial^2 f}{\partial y \partial x}$ both mean to differentiate f first with respect to y and then with respect to x.

10 pts 10. Let a particle's path be given by

$$\mathbf{r}(t) = \ln t\,\mathbf{i} + t^2\mathbf{j} + t^2\mathbf{k}.$$

Calculate the vector projection

$$\text{proj}_{\mathbf{v}(t)}\mathbf{a}(t)$$

when $t = 1$.

$$\vec{v}(t) = \vec{r}\,'(t) = \langle \frac{1}{t}, 2t, 2t \rangle$$

$$\vec{a}(t) = \vec{v}\,'(t) = \langle -\frac{1}{t^2}, 2, 2 \rangle$$

$$\text{proj}_{\vec{v}(t)}\vec{a}(t) = \frac{\vec{a}(t)\cdot\vec{v}(t)}{|\vec{v}(t)|^2}\,\vec{v}(t) =$$

$$= \frac{-\frac{1}{t^3} + 4t + 4t}{\left(\frac{1}{t^2} + 4t^2 + 4t^2\right)} \langle \frac{1}{t}, 2t, 2t \rangle$$

when $t = 1$

$$\rightarrow \frac{-1+4+4}{1+4+4} \cdot \langle 1, 2, 2 \rangle = \underline{\langle \frac{7}{9}, \frac{14}{9}, \frac{14}{9} \rangle}$$