Math 230 - Fall 2006
Solutions to Midterm Exam 2

1. (10 points) *Consider the function* $f(x,y) = 4\sin(y/2)\cos(x/2)$. *Find the linearization* $L(x,y)$ *of* f *at the point* $A = \left(\frac{\pi}{2}, -\frac{\pi}{2}\right)$.

   Solution. The linearization $L(x,y)$ of f at A is given by

   $$L(x,y) = f(A) + f_x(A)\Big(x - \frac{\pi}{2}\Big) + f_y(A)\Big(y + \frac{\pi}{2}\Big).$$

   One gets $f(A) = -2$. Since,

   $$f_x(x,y) = -2\sin(y/2)\sin(x/2), \qquad f_y(x,y) = 2\cos(y/2)\cos(x/2),$$

   it follows

   $$f_x\left(\frac{\pi}{2}, -\frac{\pi}{2}\right) = 1 \qquad f_y\left(\frac{\pi}{2}, -\frac{\pi}{2}\right) = 1.$$

   Therefore, $L(x,y) = -2 + x + y$.

2. (10 points) *Find parametric equations of the normal line to the level surface* $\cos(x + e^y) - z = 0$ *at the point* $P = (0, \ln(\pi/2), 0)$.

   Solution. Set $f(x,y,z) = \cos(x + e^y) - z$. The direction of the normal line to the level surface $f(x,y,z) = 0$ at P is given by the gradient vector $\nabla f(0, \ln(\pi/2), 0)$. But,

   $$\nabla f(x,y,z) = < -\sin(x+e^y),\ -e^y\sin(x+e^y),\ -1 > .$$

   Hence

   $$\nabla f\left(0, \ln\left(\frac{\pi}{2}\right), 0\right) = \left\langle -1, -\frac{\pi}{2}, -1 \right\rangle.$$

   Parametric equations of the normal line are:

   $$\begin{cases} x = -t \\ y = \ln(\frac{\pi}{2}) - \frac{\pi}{2}t \\ z = -t \end{cases} \qquad t \in \mathbb{R}$$

3. (10 points) *Use the Chain Rule to find* $\frac{df}{dt}$ *at* $t = \frac{\pi}{2}$ *if*

   $$f(x,y,z) = xy + y^2 z + 3zx, \quad x = \cos(t), \quad y = \sin(t), \quad z = \cos(5t).$$

Solution. One has

$$\begin{aligned}\frac{df}{dt} &= f_x\frac{dx}{dt}+f_y\frac{dx}{dt}+f_z\frac{dx}{dt}=(y+3z)(-\sin t)+(x+2yz)(\cos t)+\\ &+ (y^2+3x)(-5\sin(5t)).\end{aligned}$$

But $(x,y,z)=(0,1,0)$ at $t=\frac{\pi}{2}$, hence

$$\frac{df}{dt}\left(\frac{\pi}{2}\right)=-1-5=-6.$$

4. Suppose the temperature at the point (x,y,z) is

$$T(x,y,z)=\ln(x+2y+5z).$$

(a) (8 points) *Find the direction in which the temperature increases most rapidly at* $P=(1,5,-2)$.

Solution. The direction is given by any non-zero vector that is parallel to the gradient vector $\nabla T(P)$. But,

$$\nabla T=\left\langle\frac{1}{x+2y+5z},\frac{2}{x+2y+5z},\frac{5}{x+2y+5z}\right\rangle$$

Therefore $<1,2,5>$ gives the direction in which the temperature increases most rapidly at $P=(1,5,-2)$.

(b) (3 points) *Find the maximum rate of change of the temperature at the point* $P=(1,5,-2)$.

Solution. The maximum rate of change of the temperature at the point $P=(1,5,-2)$ is $|\nabla T(P)|=\sqrt{30}$.

5. Consider the function $f(x,y)=x^3+2y^3-3y^2-3x-12y$.

(a) (5 points) *Find all critical points of* f.

Solution. $f_x=(x,y)=3x^2-3,\qquad f_y=(x,y)=6y^2-6y-12$.
Solving the system of equations

$$\begin{cases} f_x(x,y)=0\\ f_y(x,y)=0\end{cases}\quad\Leftrightarrow\quad\begin{cases}3x^2-3=0\\ 6y^2-6y-12=0\end{cases}$$

one gets the critical points $(1,2)$, $(1,-1)$, $(-1,2)$, and $(-1,-1)$.

(b) (4 points) *Find all the second partial derivatives of f.*

Solution. $f_{xx} = 6x, \quad f_{yy} = 12y - 6, \quad f_{xy} = f_{yx} = 0.$

(c) (5 points) *Classify the critical points of f.*

Solution.

- $(1, -1)$ and $(-1, 2)$ are saddle points.
- $(1, 2)$ is a local minimum
- $(-1, -1)$ is a local maximum

6. Consider the function $f(x, y) = x^2 + \frac{1}{2}\left(y^2 + y\right)$ on the unit circle $x^2 + y^2 = 1$.

(a) (9 points) Use the method of Lagrange multipliers to find all points of the unit circle at which f has possible extreme values.

Solution. Set $g(x, y) = x^2 + y^2$ and solve the system of equations

$$\begin{cases} \nabla f = \lambda \nabla g \\ x^2 + y^2 = 1 \end{cases}$$

This gives

$$\begin{cases} 2x & = & 2\lambda x \\ y + \frac{1}{2} & = & 2\lambda y \\ x^2 + y^2 & = & 1 \end{cases}$$

- Case 1: $x = 0$ then $0^2 + y^2 = 1 \;\Rightarrow\; y = \pm 1$.
- Case 2: $x \neq 0$ then $\lambda = 1 \;\Rightarrow y + \frac{1}{2} = 2y$. One gets $y = \frac{1}{2}$ and

$$x^2 + \frac{1}{4} = 1, \quad \text{that is,} \quad x = \pm\frac{\sqrt{3}}{2}.$$

The solutions are $(0, 1)$, $(0, -1)$, $(\sqrt{3}/2, 1/2)$, and $(-\sqrt{3}/2, 1/2)$.

(b) (2 points) *Find the maximum and minimum values of f.*

Solution.

$$f(0, 1) = 1, \quad f(0, -1) = 0, \quad f\left(\pm\frac{\sqrt{3}}{2}, \frac{1}{2}\right) = \frac{9}{8}.$$

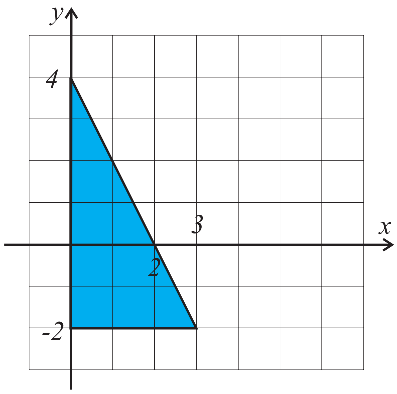
The maximum value is $M = 9/8$ and minimum value is $m = 0$.

7. Consider the double integral $\displaystyle\int_0^3 \int_{y=-2}^{y=4-2x} x \; dy \, dx.$

(a) (3 points) *Sketch the region of integration*

Solution.

The region of integration is the **interior** of the triangle:



(b) (5 points) *Evaluate the double integral.*

Solution.

$$\int_0^3 \int_{y=-2}^{y=4-2x} x \; dy \, dx = \int_0^3 x(6-2x) \; dx = \left[3x^2 - \frac{2}{3}x^3\right]_0^3 = 9.$$

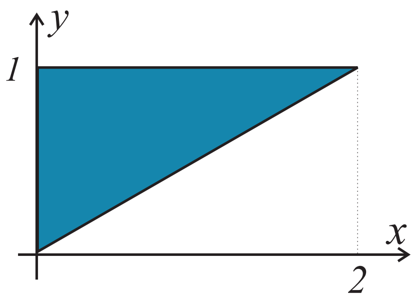
(c) (5 points) *Write an equivalent integral by reversing the order of integration.*

Solution. $\displaystyle\int_{-2}^4 \int_{x=0}^{x=2-\frac{y}{2}} x \; dx \, dy$

8. (10 points) *Find the area of the part of the surface* $z = 3x + y^2$ *that lies above the triangular region in the* xy*-plane with vertices* $(0,0)$, $(2,1)$, *and* $(0,1)$. *Sketch the triangular region.*

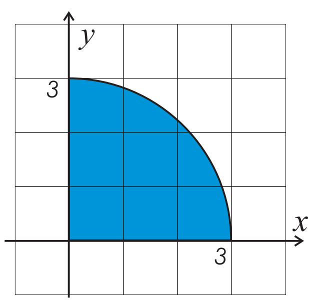
Solution. Let $z = f(x,y) = 3x + y^2$

$$\begin{aligned} A(S) &= \int_D \sqrt{(f_x)^2 + (f_y)^2 + 1} \; dA = \int_0^1 \int_0^{2y} \sqrt{10 + 4y^2} \; dx \, dy = \\ &= \frac{1}{6}\left(14^{3/2} - 10^{3/2}\right). \end{aligned}$$



9. Consider $\int_0^3 \int_0^{\sqrt{9-x^2}} (x^2+y^2)\ dy\ dx.$

(a) (3 points)*Sketch the region of integration for the double integral.*



(b) (8 points) Use polar coordinates to evaluate the double integral.

Solution.

$$\int_0^3 \int_0^{\sqrt{9-x^2}} (x^2+y^2)\ dy\ dx = \int_o^{\frac{\pi}{2}} \int_0^3 r^2\, r\ dr\, d\theta = \frac{81\pi}{8}.$$