1. (10 points) Show that the 4 points $P_1(0,0,0)$, $P_2(2,3,0)$, $P_3(1,-1,1)$, $P_4(1,4,-1)$ are coplanar (they lie on the same plane), and find the equation of the plane that contains them.

Compute the volume of the parallelepiped defined by

$$\overrightarrow{P_1P_2} = (2,3,0),\ \overrightarrow{P_1P_3} = (1,-1,1),$$
$$\overrightarrow{P_1P_4} = (1,4,-1):$$

$$\left|\begin{vmatrix} 2 & 1 & 1 \\ 3 & -1 & 4 \\ 0 & 1 & -1 \end{vmatrix}\right| = \left|2(1-4) - 3(-1-1)\right| = -6+6=0$$

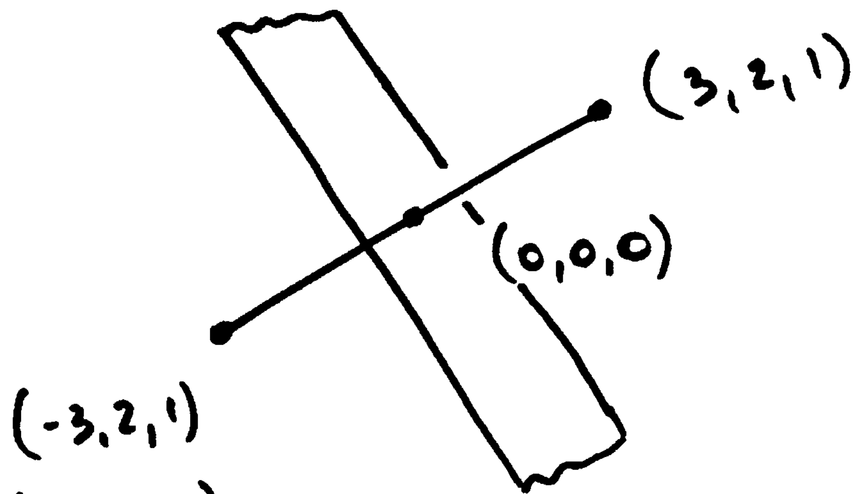
Parallelepiped is flat.

$$\vec{n} = \begin{vmatrix} i & 2 & 1 \\ j & 3 & -1 \\ k & 0 & 1 \end{vmatrix} = i\cdot 3 - j\,2 + k(-2-3) = \begin{bmatrix} 3 \\ -2 \\ -5 \end{bmatrix}$$

Plane

$$3x - 2y - 5z = 0$$

2. (10 points) Find the equation of the plane that is equidistant from the points (3, 2, 1) and (−3, −2, −1) (that is, every point on the plane has the same distance from the two given points).



$P = (0,0,0)$

$\vec{n} = (3,2,1)$

Plane through the midpoint P with normal $\vec{n}$ is

$$\boxed{3x + 2y + z = 0}$$

3. (6 points) Find the vector projection of $\vec{b}$ onto $\vec{a}$, if $\vec{a} = < 4, 2, 0 >$ and $\vec{b} = < 1, 1, 1, >$.

$$\mathrm{Proj}_{\vec{a}}\,\vec{b} = \frac{\vec{b}\cdot\vec{a}}{|\vec{a}|}\,\frac{\vec{a}}{|\vec{a}|} = \frac{\vec{b}\cdot\vec{a}}{|\vec{a}|^2}\,\vec{a} =$$

$$= \frac{\begin{bmatrix}1\\1\\1\end{bmatrix}\cdot\begin{bmatrix}4\\2\\0\end{bmatrix}}{4^2+2^2+0^2}\begin{bmatrix}4\\2\\0\end{bmatrix} = \frac{6}{20}\begin{bmatrix}4\\2\\0\end{bmatrix} = \frac{3}{10}\begin{bmatrix}4\\2\\0\end{bmatrix}$$

4. (12 points) Consider the curve $\vec{r}(t) = \sqrt{2}\cos t\,\vec{i} + \sin t\,\vec{j} + \sin t\,\vec{k}$.
   (a) (8 points) Find the **unit tangent** vector function $\vec{T}(t)$ and the **unit normal** vector function $\vec{N}(t)$.
   (b) (4 points) Compute the curvature κ.

$$\vec{r}'(t) = \begin{bmatrix} -\sqrt{2}\sin t \\ \cos t \\ \cos t \end{bmatrix}; \quad \vec{r}''(t) = \begin{bmatrix} -\sqrt{2}\cos t \\ -\sin t \\ -\sin t \end{bmatrix}$$

$$|\vec{r}'(t)| = \sqrt{(-\sqrt{2}\sin t)^2 + \cos^2 t + \cos^2 t} = \sqrt{2}.$$

$$\vec{r}'(t) \times \vec{r}''(t) = \begin{vmatrix} i & -\sqrt{2}\sin t & -\sqrt{2}\cos t \\ j & \cos t & -\sin t \\ k & \cos t & -\sin t \end{vmatrix} =$$

$$= i(-\sin t\cos t + \sin t\cos t) - j(\sqrt{2}\sin^2 t + \sqrt{2}\cos^2 t) + k(\sqrt{2}\sin^2 t + \sqrt{2}\cos^2 t) =$$

$$= i\,0 - j\sqrt{2} + k\sqrt{2}.$$

$$|\vec{r}'(t) \times \vec{r}''(t)| = \sqrt{2+2} = \sqrt{4} = 2.$$

a) $$\vec{T}(t) = \frac{\vec{r}'(t)}{|\vec{r}'(t)|} = \frac{1}{\sqrt{2}}\begin{bmatrix} -\sqrt{2}\sin t \\ \cos t \\ \cos t \end{bmatrix} = \begin{bmatrix} -\sin t \\ \frac{1}{\sqrt{2}}\cos t \\ \frac{1}{\sqrt{2}}\cos t \end{bmatrix}.$$

$$\vec{T}'(t) = \begin{bmatrix} -\cos t \\ -\frac{1}{\sqrt{2}}\sin t \\ -\frac{1}{\sqrt{2}}\sin t \end{bmatrix}, \quad |\vec{T}'(t)| = \sqrt{\cos^2 t + \tfrac{1}{2}\sin^2 t + \tfrac{1}{2}\sin^2 t} = 1$$

$$\vec{N}(t) = \frac{\vec{T}'(t)}{|\vec{T}'(t)|} = \begin{bmatrix} -\cos t \\ -\frac{1}{\sqrt{2}}\sin t \\ -\frac{1}{\sqrt{2}}\sin t \end{bmatrix}$$

b) $$\boxed{\kappa = \frac{|\vec{r}'(t) \times \vec{r}''(t)|}{|\vec{r}'(t)|^3} = \frac{2}{2^{3/2}} = \frac{1}{\sqrt{2}}.}$$

5. (10 points) Find the length of the curve with parametric equation:

$$\vec{r}(t) = < e^t, e^t \sin t, e^t \cos t >.$$

between the points $(1, 0, 1)$ and $(e^{2\pi}, 0, e^{2\pi})$.

$$\vec{r}(t_1) = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \Longrightarrow t_1 = 0$$

$$\vec{r}(t_2) = \begin{bmatrix} e^{2\pi} \\ 0 \\ e^{2\pi} \end{bmatrix} \Longrightarrow t_2 = 2\pi$$

$$\vec{r}\,'(t) = \begin{bmatrix} e^t \\ e^t \sin t + e^t \cos t \\ e^t \cos t - e^t \sin t \end{bmatrix} = e^t \begin{bmatrix} 1 \\ \sin t + \cos t \\ \cos t - \sin t \end{bmatrix}$$

$$\boxed{L} = \int_0^{2\pi} |\vec{r}\,'(t)|\, dt = \int_0^{2\pi} e^t \sqrt{1^2 + (\sin t + \cos t)^2 + (\cos t - \sin t)^2}\, dt =$$

$$= \int_0^{2\pi} e^t \sqrt{1^2 + \sin^2 t + \underline{2 \sin t \cos t} + \cos^2 t + \sin^2 t - \underline{2 \sin t \cos t} + \cos^2 t}\, dt =$$

$$= \int_0^{2\pi} e^t \sqrt{3}\, dt = \sqrt{3}\, e^t \Big|_0^{2\pi} =$$

$$= \boxed{\sqrt{3}\,(e^{2\pi} - 1)}$$

6. (12 points) A spaceship is traveling with acceleration

$$\vec{a}(t) = < e^t, t, \sin 2t > .$$

At $t = 0$, the spaceship was at the origin, $\vec{r}(0) = < 0, 0, 0, >$, and had initial velocity $\vec{v}(0) = < 1, 0, 0 >$. Find the position of the spaceship at $t = \pi$.

$$\vec{v}(t) = \vec{v}(0) + \int_0^t \vec{a}(\tau)\, d\tau =$$

$$= \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \int_0^t \begin{bmatrix} e^\tau \\ \tau \\ \sin 2\tau \end{bmatrix} d\tau =$$

$$= \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} e^t - 1 \\ \frac{1}{2}t^2 \\ \frac{1}{2}(1-\cos(2t)) \end{bmatrix} = \begin{bmatrix} e^t \\ \frac{1}{2}t^2 \\ \frac{1}{2}(1-\cos(2t)) \end{bmatrix}$$

$$\vec{r}(t) = \vec{r}(0) + \int_0^t \vec{v}(\tau)\, d\tau =$$

$$= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + \int_0^t \begin{bmatrix} e^\tau \\ \frac{1}{2}\tau^2 \\ \frac{1}{2}(1-\cos(2\tau)) \end{bmatrix} d\tau =$$

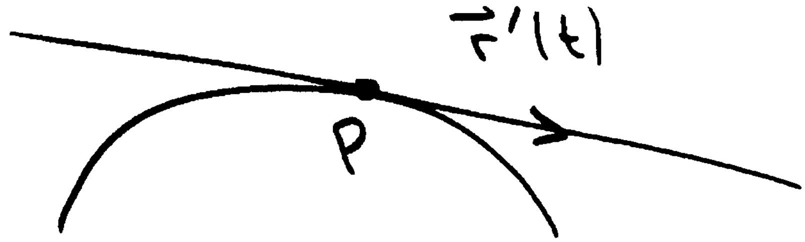
$$\left.\begin{bmatrix} e^\tau \\ \frac{1}{6}\tau^3 \\ \frac{1}{2}\tau - \frac{1}{4}\sin(2\tau) \end{bmatrix}\right|_0^t = \begin{bmatrix} e^t - 1 \\ \frac{1}{6}t^3 \\ \frac{1}{2}t - \frac{1}{4}\sin(2t) \end{bmatrix}$$

$$\vec{r}(\pi) = \begin{bmatrix} e^\pi - 1 \\ \frac{1}{6}\pi^3 \\ \frac{\pi}{2} - 0 \end{bmatrix}$$

7. (10 points) Write the equation of the tangent line to the curve with parametric equation:

$$\vec{r}(t) = < \sqrt{t}, 1, t^4 >,$$

at the point (1, 1, 1).



$t_0 = 1$

$\vec{r}(t_0) = (1, 1, 1) = P$

Directional vector

$$\vec{r}'(t_0) = \vec{r}'(1) = \begin{bmatrix} \frac{1}{2}\frac{1}{\sqrt{t_0}} \\ 0 \\ 4t_0^3 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ 0 \\ 4 \end{bmatrix}$$

$$\underbrace{\begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + t \begin{bmatrix} 1/2 \\ 0 \\ 4 \end{bmatrix}}$$

8. (12 points) Using cylindrical coordinates, find the parametric equations of the curve that is the intersection of the cylinder $x^2 + y^2 = 4$ and the cone $z = \sqrt{x^2 + y^2}$.

<u>Eq 1:</u> $r^2 = 4$

<u>Eq 2:</u> $z = r$

$$\begin{cases} r(t) = 2 \\ \theta(t) = t \\ z(t) = 2 \end{cases}$$