Differential rigidity of group actions

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Main References

- A. Katok and R. Spatzier, *Differential rigidity of Anosov actions of Hyperbolic Abelian groups*, preprint,1992.
- A. Katok and R. Spatzier, *First cohomology of Anosov actions of higher rank Abelian groups and applications to rigidity*, Inst. Hautes Etudes Sci. Publ. Math. **79**(1994), 131–156.
- A. Katok and R. Spatzier, *Differential rigidity of Anosov actions of higher rank Abelian groups and algebraic lattice actions*, Dynamical Systems and Related Topics (Volume dedicated to D.V. Anosov), Proc. Steklov Math.Inst., Vol. 216 (1997),287–314

Last two papers are available on A.Katok's site: www.math.psu.edu/katok_a/papers.html

Hyperbolic Fixed Points

Definition 1. A linear map $L : \mathbb{R}^n \to \mathbb{R}^n$ is called $h$yperbolic if absolute values of all eigenvalues are different from 1.

Definition 2. Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be C^∞ function, such that x_0 is a fixed point. We say that x_0 is a $h$yperbolic fixed point if the derivative at the point x_0 is a hyperbolic linear map.

One dimensional case

Example 1. *Let $f : \mathbb{R} \to \mathbb{R}$ be C^∞ function.* $f(0) = 0$.

Denote $f'(0) = a$. Then $f(x) = ax + o(x)$.

If $|a| < 1$ then $f(x)$ is a contracting map in some neighborhood of 0. If $|a| > 1$ then $f(x)$ is an expanding map in some neighborhood of 0.

If $|a| = 1$ then $o(x)$ determines the behavior of $f^n(x)$.

If $|f(x) - g(x)| < \epsilon$ and $|f'(x) - g'(x)| < \epsilon$, then the function g has a hyperbolic fixed point x_0 close to 0, and this is the only fixed point of g close to 0.

Hyperbolic Linear Maps

Example 2. *Let $L : \mathbb{R}^2 \to \mathbb{R}^2$ be a matrix*

$$L = \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}$$

where $0 < \lambda < 1 < \mu$.

$\mathbb{R}^2$ is a direct sum of two invariant subspaces: $\mathbb{R}^2 = E^- \oplus E^+$.

There exists a metric on $\mathbb{R}^2$, such that L is a contracting map on E^- and an expanding map on E^+.

$E^- = \{v | L^n(v) \to 0\}$ and $E^+ = \{v | L^{-n}(v) \to 0\}$.

All this true for any hyperbolic linear map L.

Hadamard-Perron Theorem

Theorem 0.1 (Hadamard-Perron Theorem)**.** *Let 0 be a hyperbolic fixed point of an invertible C^∞ map $f : \mathbb{R}^n \to \mathbb{R}^n$. We denote as L the derivative of f at* 0. *As we know $L = E^- \oplus E^+$. There exists a small ball B around 0, such that the sets*

$$W^s = \{x | f^n(x) \in B \text{ for all } n \geq 0 \quad \text{and} \lim_{n \to +\infty} f^n(x) = 0\}$$
$$W^u = \{x | f^{-n}(x) \in B \text{ for all } n \geq 0 \quad \text{and} \lim_{n \to +\infty} f^{-n}(x) = 0\}$$

are C^∞ manifolds and their tangent spaces at 0 are E^- and E^+. There exists a Riemannian metric such that the map f is a contracting map on W^s and expanding map on W^u. There exist λ, $0 < \lambda < 1$ and C, such that $dist(f^n(x), f^n(y)) \leq C\lambda^n$ for all $n \geq 0$ and $x, y \in W^s$ $dist(f^{-n}(x), f^{-n}(y)) \leq C\lambda^n$ for all $n \geq 0$ and $x, y \in W^u$

Hartmann-Grobman Theorem

Let the map f has a hyperbolic fixed point p. Then:

- Small C^1 perturbation of f does not destroy the fixed point, but only moves it to a close position.
- The map f has two nice manifolds intersecting at the fixed points that are small perturbations of $E^{\pm}$.
- Let f and g are C^1 close. Denote the hyperbolic point close to p as q, then there are neighborhoods B_p and B_q of p and q and a homeomorphism $h : B_p \to B_q$, such that $f = h^{-1} \circ g \circ h$

1 Anosov maps

Global behavior

Let M be a smooth manifold and $f : M \to M$ a smooth diffeomorphism. We denote as

- T_xM(or T_x the tangent space at the point x,
- TM the tangent bundle to the manifold M,
- $Df_x : T_x \to T_{f(x)}$ the derivative of the map f at the point x,
- $||v||$ the norm of a tangent vector $v \in T_xM$ with respect to the chosen Riemannian metric.

We say that a subbundle $E = \cup_{x\in M} E_x$ is invariant under Df if

$$Df_x(E_x) \subset E_{f(x)}.$$

Definition of Anosov map

Definition 3. We say that a smooth diffeomorphism $f : M \to M$ is Anosov, if M is compact and there exist $\lambda, 0 < \lambda < 1$, $C > 0$ and a direct sum decomposition of the tangent bundle of M in two invariant continuous subbundles , $TM = E^- \oplus E^+$, such that: for each integer $m > 0$ and $v \in E_x^-$:

$$||Df_x^m v|| \leq C\lambda^m ||v||$$

for each integer $m > 0$ and $v \in E_x^+$:

$$||Df_x^{-m} v|| \leq C\lambda^m ||v||$$

Example of Anosov map

Let $M = \mathbb{R}^2/\mathbb{Z}^2$ and $A \in GL(2, \mathbb{Z})$.

For example, let us take

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$$

. Eigenvalues are

$$\lambda_{1,2} = \frac{3 \pm \sqrt{5}}{2}$$

Stability of Anosov maps

Theorem 1.1 (Anosov). *If a map f is Anosov and a map g is obtained by a small in C^1 topology perturbation, then g is also Anosov.*

Theorem 1.2 (Anosov). *Anosov maps are structurally stable. It means that if f is an Anosov map, then there exists a C^1 neighborhood of f, such that for each map g from this neighborhood there exists a homeomorphism $h : M \to M$ and $f = h^{-1} \circ g \circ h$*

Remarks

Remark 1. It is easy to see that h is very rarely C^1 .

Remark 2. In the definition of the Anosov map we require the subbbundles $E_x^{\pm}$ to depend on x only continuously. It is easy to prove that they are always depend on x Holder-continuously, and usually they are not even C^1. In the example, that we saw they were C^∞. This is an exception.

2 Stable and Unstable manifolds

Local Stable manifolds

Let $f : M \to M$ be an Anosov map. There exists a family of balls $B(p)$ with radius continuously depending on a point $p \in M$ such that the sets $W_p^{s,loc}, W_p^{u,loc}$

$$\{x | f^n(x) \in B_{f^n(p)} \text{ for } n \geq 0, \lim_{n\to+\infty} dist(f^n(x), f^n(p) = 0\}$$

$$\{x | f^{-n}(x) \in B_{f^n(p)} \text{ for } n \geq 0, \lim_{n\to+\infty} dist(f^{-n}(x), f^{-n}(p) = 0\}$$

are C^∞ manifolds and their tangent spaces at p are E_p^- and E_p^+.

$$f(W_p^{s,loc}) \subset W_{f(p)}^{s,loc}$$

$$f^{-1}(W_p^{u,loc}) \subset W_{f^{-1}(p)}^{u,loc}$$

There exists a Riemannian metric such that the map f is a contracting map on W_p^s and expanding map on W^u.

Local Stable Manifolds

Let $f : M \to M$ be an Anosov map. There exists a family of balls $B(p)$ with radius continuously depending on a point $p \in M$ such that the sets $W_p^{s,loc}, W_p^{u,loc}$

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$$f(W_p^{s,loc}) \subset W_{f(p)}^{s,loc}$$

$$f^{-1}(W_p^{u,loc}) \subset W_{f^{-1}(p)}^{u,loc}$$

There exist λ, $0 < \lambda < 1$ and C, such that $dist(f^n(x), f^n(y)) \leq C\lambda^n$ for all $n \geq 0$ and $x, y \in W_p^{s,loc}$ $dist(f^{-n}(x), f^{-n}(y)) \leq C\lambda^n$ for all $n \geq 0$ and $x, y \in W_p^{u,loc}$

Global Stable and Unstable Manifolds

We can define global stable manifolds as

$$W_p^s = \{x \in M| \lim_{n\to+\infty} dist(f^n(x), f^n(p) = 0\} =$$

$$\cup_{n=0}^{\infty} W_{f^{-n}(p)}^{s,loc}$$

The same definition holds for W_p^u, after replacing f with f^{-1}. Properties of Global Manifolds:

1. W^s and W^u are f invariant.
2. $T_x W_x^s = E_x^-$ and $T_x W_x^u = E_x^+$.
3. If $W_x^s \cap W_y^s$ is not empty, then $W_x^s = W_y^s$. The same for W_x^u, W_y^s.

3 Anosov Flows

Smooth Flows

We say that a family $\{f_t|t \in \mathbb{R}\}$ of smooth diffeomorfisms $f_t : M \to M$ is a flow if

1. $f_0(x) = x$.
2. $f_{t+s} = f_t \circ f_s$.
3. $F(t, x) = f_t(x)$ is a smooth function.

If we denote $\frac{df_t(x)}{dt}|_{t=0} = v(x)$, then $v(x)$ is a smooth vector field and $f_t(x_0)$ is a solution of ordinary differential equation

$$\frac{dx}{dt} = v(x), x(0) = x_0.$$

Anosov flow

A smooth flow f_t on a compact C^∞ manifold M is called Anosov, if the tangent bundle is a direct sum of three subbundles $TM = E^- \oplus E^0 \oplus E^+$, where E^0_x is the tangent line to the orbit of x, and $E^\pm_x$ satisfy the same properties as for Anosov maps, there exist $\lambda, 0 < \lambda < 1$, $C > 0$ such that: for each $t \geq 0$ and $v \in E^-_x$:

$$||Df_t(v)|| \leq C\lambda^t ||v||$$

for each $t \geq 0$ and $v \in E^+_x$:

$$||Df_{-t}(v)|| \leq C\lambda^t ||v||$$

Anosov flow

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PHD

Let M be a smooth manifold and $f : M \to M$ be a C^∞ map. Assume its tangent bundle could be decomposed in the direct sum of invariant continuous subbundles

$$TM = E^- \oplus E^0 \oplus E^+$$

and there exist numbers $\lambda < 1 < \mu$ and $\epsilon > 0$ such that

$$\overline{\lim_{n\to\infty}} \sqrt[n]{||Df^n(v)||} \leq \lambda, \text{ if } v \in E^-$$

$$\lambda + \epsilon \leq \underline{\lim_{n\to\infty}} \sqrt[n]{||Df^n(v)||} \text{ and } \overline{\lim_{n\to\infty}} \sqrt[n]{||Df^n(v)||} \leq \mu - \epsilon \text{ if } v \in E^0$$

$$\underline{\lim_{n\to\infty}} \sqrt[n]{||Df^n(v)||} \geq \mu \text{ if } v \in E^+$$

Questions

We want to answer the following questions:

1. Is a C^1 close PHD also PHD?

2. What foliations we have for PHD?

3. Is PHD structurally stable ?

Persistence of invariant distributions

Let M be a smooth manifold and $f : M \to M$ be a C^1 map. Assume its tangent bundle could be decomposed in the direct sum of invariant continuous subbundles

$$TM = E^1 \oplus E^2 \oplus \cdots \oplus E^k$$

and there exist numbers

$$\lambda_1 \leq \mu_1 < \lambda_2 \leq \mu_2 < \cdots < \lambda_k \leq \mu_k, c, C$$

such that if $v \in E^i$ then

$$\lambda_i \leq \varliminf_{n\to\infty} \sqrt[n]{||Df^n(v)||} \text{ and } \varlimsup_{n\to\infty} \sqrt[n]{||Df^n(v)||} \leq \mu_i$$

If g is C^1 closed to f then g has invariant continuos subbundles V^i that are C^0 close to E^i.

Perturbation of Anosov flows

We say that two smooth flows are C^1 close if their vector fields are C^1 close. It is equivalent to saying that two smooth flows f_t and g_t are C^1 close if f_t is C^1 close to g_t as a map, for all $0 \leq t \leq 1$.

Exercise: If f_t is an Anosov flow, then there is a C^1 neighborhood of f_t such that all smooth flows from this neighborhood are Anosov.

Integrability of invariant distributions

Let M be a smooth manifold and $f : M \to M$ be a C^∞ map. Assume its tangent bundle could be decomposed in the direct sum of invariant continuous subbundles $TM = E^- \oplus E^+$ and there exist numbers $\lambda < \mu$, such that

$$\varlimsup_{n\to\infty} \sqrt[n]{||Df^n(v)||} \leq \lambda, \text{ if } v \in E^- \text{ and } \varliminf_{n\to\infty} \sqrt[n]{||Df^n(v)||} \geq \mu$$

Assume that $\lambda < 1$. Then there exists a continuous invariant foliation W^s with C^1 leaves and $TW^s = E^-$. Moreover,

$$W^s_p = \{x, \lim_{n\to\infty} \sqrt[n]{dist(f^n(p), f^n(x))} = 0\}$$

If in addition $\mu \geq 1$, then the leaves of the foliation are C^∞.

Foliations

Anosov flow:

$$TM = E^- \oplus E^0 \oplus E^+$$

There are several foliations related to an Anosov flow.

- Orbit foliation. It is smooth.
- Strong stable and unstable foliations. Only continuous.
- Weak stable and unstable foliations. Only continuous.

Stability of Anosov Flows

Let f_t be Anosov flow on a smooth, compact manifold M, and g_t is C^1 close smooth flow. There exists a unique continuous homeomorphism $h : M \to M$ which is C^0 close to the Id map, such that orbits of the flow f_t are mapped to the orbits of the flow g_t:

$$f_t(x) = h^{-1} \circ g_{s(t,x)} \circ h(x)$$

$s(t, x)$ has a property that

$$s(t_1 + t_2, x) = s(t_1, x) + s(t_2, f_{t_1}(x)) \text{ (cocycle).}$$

Structural stability for some PHD

M, f and $TM = E^- \oplus E^0 \oplus E^+$, where

- E^- is contracting
- E^0 has zero growth
- E^+ is expanding
- Assume that E^0 is integrable and E^0 is C^1 . If g is sufficiently C^1 close to f then:

1. It has central, weak stable(unstable) and strong stable(unstable) foliations
2. There is a homeomorphism h that maps central foliation to the central and is uniquely defined in the transversal direction. It could be chosen to be smooth along the central foliation and C^0 close to the identity.

Examples of Anosov Flows:Suspension

Let M be a smooth compact manifold, f be an Anosov map on M. We construct an Anosov flow on a new manifold N. First we define a flow F_t on the manifold $rn \times M$.

$$F_t((s, x)) = (s + t, x)$$

Then we identify points $(t + n, x)$ and $(t, f^n(x)$. The resulting manifold we call N, The resulting flow $f(t)$. Exercise: Prove that if f was Anosov, then f_t is Anosov.

Geodesic flow

Let Γ be a cocompact lattice of $SL(2, R)$,

$$A_t = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}$$

We define the flow f_t on the manifold $SL(2, R)/\Gamma$ as

$$f_t(X) = A_t X$$

Exercise: Prove that it is Anosov.

Higher rank actions

We say that an action of $\mathbb{Z}^k$ is Anosov if there is an Anosov element.We say that $\mathbb{R}^k$ action is Anosov if there is a PHD element with central distribution which is tangent bundle for the orbit foliation.

Example of Anosov $\mathbb{R}^k$ action

Consider two commuting matrices A and B on $\mathbb{T}^3$. Construct a new manifold $\mathbb{T}^3 \times \mathbb{R}^2$. Define an $\mathbb{R}^2$ action on this manifold as

$$F_{t_1 t_2}((x, s_1, s_2)) = (x, t_1 + s_1, t_2 + s_2)$$

We define a $\mathbb{Z}^2$ action as

$$(x, s + n, t + m) = (A^n B^m x, s, t)$$

F commutes with this this action, hence it is well defined on the orbits of this action.

Algebraic $\mathbb{R}^k$ actions

Let H be a connected Lie group (say $SL(n+1, \mathbb{R}$ or $SL(n+1, \mathbb{C})$and there is a subgroup of H isomorphic to $\mathbb{R}^k$ (say diagonal matrices with real coefficients).Assume Γ is cocompact lattice and C is a compact subgroup commuting with it. Then the $\mathbb{R}^k$ action on H/Γ descends to an action on $C\backslash H/\Gamma$. The general algebraic action is a finite factor of such an action.The linear part of algebraic $\mathbb{R}^k$ action is adjoint representation of this action on $LA(C)\backslash LA(H)$.

C^1 closeness

We say that two $\mathbb{Z}^k$ actions are C^1 close if they are C^1 close on a set of generators.We say that two $\mathbb{R}^k$ actions are C^1 close if they are C^1 close on a compact set that generate the whole group. We say two foliations are C^1 close if their tangent distributions are C^1 close.

Main Theorem

Theorem 3.1. *Let α be an algebraic Anosov $\mathbb{R}^k$ action on a manifold M, for $k \geq 2$, such that the linear part is semisimple. Assume that for any maximal nontrivial intersection $\cap_{i=1,\ldots,r} W^s_{b_i}$ of stable manifolds of elements $b_1, \ldots, b_r \in \mathbb{R}^k$ there exists an element $a \in \mathbb{R}^k$ such that for a.e $x \in M$, $\cap_{i=1,\ldots,r} E^-_{b_i} \subset E^0_a$ and such that a.e leaf of $\cap_{i=1,\ldots,r} W^s_{b_i}$ is contained in the ergodic component of the one parameter subgroup ta (with respect to Haar measure). Then the C^1 perturbation is C^∞ conjugated to the the α by map close to Id. In fact even C^1-small perturbation of its orbit foliation is C^∞ conjugated to the the α by map close to Id.*

Smooth conjugation to a linear map

Let 0 be a hyperbolic fixed point for $f:\mathbb{R}^n \to \mathbb{R}^n$. When it is C^∞ conjugated to its linear part?

$$f = h^{-1} \circ L \circ h$$

Assume $f = (f_1, f_2, ..., f_n), h = (h_1, h_2, ..., h_n)$ and L is a diagonal matrix with λ_i on the diagonal.Write down the Taylor series for f_i and h_i

$$f_i = \sum_{k\in N^n} f_{i,k}x^k = \lambda_i x_i + \sum_{|k|>0} f_{i,k}x^k$$

$$h_i = \sum_{k\in N^n} h_{i,k}x^k = x_i + \sum_{|k|>0} h_{i,k}x^k$$

Formal conjugation

We want formally to solve:

$$h(f(x)) = L(h(x))$$

Left side equals

$$\sum_{k\in N^n} h_{i,k}(f(x))^k = \lambda_i h_i$$

$$\lambda_i f_i + \sum_{|k|>0} h_{i,k}(f_{1,k})^{k_1} \cdots (f_{n,k})^{k_n} =$$

$$\lambda_i f_i + \sum_{|k|>1} h_{i,k}(\lambda_1 x_1 + \sum_{|l_1|>1} f_{1,l_1}x^{l_1})^{k_1} \cdots (\lambda_n x_n + \sum_{|l_n|>1} f_{n,l_n}x^{l_n})^{k_n}$$

Quadratic term on the left side is:

$$\sum h_{i,k}\lambda^k x^k + \text{quadratic from f}$$

on the right side it is:

$$\sum h_{i,k}\lambda_i x^k$$

$$h_{i,k}(\lambda_i - (\lambda)^k) = known$$

$$h_{i,k}\lambda^k x^k + \text{ known terms} = h_{i,k}\lambda_i x^k$$

The relation $\lambda_i = \lambda_1^{k_1}\lambda_2^{k_2}\cdots\lambda_n^{k_n}$ we call a resonance.Poincare: no resonance implies existance of formal h.

4 Normal forms, stationery case

Formal change

f has a hyperbolic fixed point at 0, $L = Df(0)$. We want to find h in a form of formal power series such that

$$L = h \circ f \circ h^{-1}$$

or more generally

$$N = h^{-1} \circ f \circ h$$

Assume $f = (f_1, f_2, ..., f_n), h = (h_1, h_2, ..., h_n)$ and L is a diagonal matrix with λ_i on the diagonal.Where f_i and h_i are

$$f_i = \sum_{k \in N^n} f_{i,k} x^{\mathbf{k}} = \lambda_i x_i + \sum_{|k|>0} f_{i,k} x^{\mathbf{k}}$$

$$h_i = \sum_{k \in N^n} h_{i,k} x^{\mathbf{k}} = x_i + \sum_{|k|>0} h_{i,k} x^{\mathbf{k}}$$

Formal change 2

Rewrite:

$$h \circ f = N \circ h$$

Assume that we found all coefficients of h for terms with degree $< l$. Look what is coefficient next to ${x_1}^{k_1} x_2^{k_2} \cdots x_n^{k_n}$ with $\sum k_i = l$. It is easy to see that it looks like

$$\lambda_1^{k_1} \lambda_2^{k_2} \cdots \lambda_n^{k_n} h_{i,k} + \text{known} =$$
$$\lambda_i h_{i,k} + n_{i,k} + \text{known}$$

Resonances

$\mathbf{k}$=$(k_1, k_2, \ldots, k_n)$ We call a pair $(i, \mathbf{k})$ a resonance relation, if

$$\lambda_i = \lambda_1^{k_1} \lambda_2^{k_2} \cdots \lambda_n^{k_n}$$

Normal form

We say that a formal power series

$$N = (p_1(x), p_2(x), \cdots, p_n(x))$$

is a normal form for the map f if

$$p_i = \sum_{\mathbf{k}} p_{i,\mathbf{k}} x^{\mathbf{k}} = \sum_{\mathbf{k}} p_{i,\mathbf{k}} x_1^{k_1} x_2^{k_2} \cdots x_n^{k_n}$$

and $p_{i,\mathbf{k}} \neq 0$, only when $(i, \mathbf{k})$ is a resonance. A monom $m = x_1^{k_1} x_2^{k_2} \cdots x_n^{k_n}$ is allowed in normal form for i-th coordinate if only if

$$m(\lambda_1, \lambda_2, \ldots, \lambda_n) = \lambda_i$$

Example

$$L = \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{4} & 0 \\ 0 & 0 & \frac{1}{8} \end{pmatrix}$$

Normal form

$$N(x,y,z) = \begin{pmatrix} *x \\ *y + *x^2 \\ *z + *xy + *x^3 \end{pmatrix}$$

Group

Theorem 4.1. *Fix $\lambda = (\lambda_1, \lambda_2, \dots \lambda_n)$. Then, normal forms with invertible linear part form a group.*

Proof: It is closed under operation of composition. Let $p(x)$ and $s(x)$ are normal forms, then

$$p_i(s(x)) = p_i(s_1(x), s_2(x), \dots, s_n(x)) =$$
$$\sum_{\mathbf{k}} p_{i,\mathbf{k}} s_1^{k_1} s_2^{k_2} \cdots s_n^{k_n}$$

Proof

We need to check that all monoms for

$$s_1^{k_1} s_2^{k_2} \cdots s_n^{k_n} = \left(\sum\right)^{k_1} \left(\sum\right)^{k_2} \dots \left(\sum\right)^{k_n}$$

are allowed in normal form.

Each monom looks like:

$$\underbrace{m_1(x) \cdot m_2(x) \cdot \dots \cdot m_{k_1}(x)}_{\text{monoms from } s_1} \cdot \dots \cdot \underbrace{u_1(x) \cdot u_2(x) \cdot \dots \cdot u_{k_n}(x)}_{\text{monoms from } s_n}$$

If we calculate it at $(\lambda_1, \lambda_2, \dots, \lambda_n)$ we get:

$$\lambda_1^{k_1} \lambda_2^{k_2} \cdots \lambda_n^{k_n} = \lambda_i$$

Inverse

To construct the inverse of normal form p we find a normal form s such that $p \circ s = Id$. We find s as a product:

$$s = r \circ t \cdots \circ u$$

We take $r = L^{-1}$ then

$$p \circ r = Id + P_2 + P_{>2}$$

Take $t = Id - P_2$ then

$$p \circ r \circ t = Id + P'_{>3}$$

Sternberg-Chen

Let f be C^{∞}map with a hyperbolic fixed point at 0. Then it could be conjugated to a normal form (with finite number of terms).

5 Non-stationery normal forms

Non-stationery normal forms

Let X be a compact, connected metric space, $f : X \to X$ be a homeomorphism (continuous dynamical system),V be a vector bundle over X, $\pi : V \to X$ be a projection. We call a function $\mathcal{F} : V \to V$ an extension of the function f if the following equality holds:

$$f \circ \pi = \pi \circ F$$

We say that an extension $\mathcal{F}$ is linear(polynomial) if it acts as a linear(polynomial) map on each fiber.

Contraction

We say that a linear extension F is a (λ, μ)-*contraction* if vectors $\lambda = (\lambda_1, \dots, \lambda_l)$ and $\mu = (\mu_1, \dots, \mu_l)$ satisfy the following conditions:

$$\lambda_1 \le \mu_1 < \lambda_2 \le \mu_2 < \cdots < \lambda_l \le \mu_l < 0.$$

and the bundle V splits into the direct sum of F–invariant sub-bundles $V_1, \dots, V_l$, continuously depending on x and for any $\epsilon > 0$ we can find a continuous family of inner products in the fibers of V such that for any vector $u \in V_i$

$$\exp(\lambda_i - \epsilon)\|u\| \le \|Fu\| \le \exp(\mu_i + \epsilon)\|u\|$$

Contraction

In the other words, a linear extension F is a (λ, μ)-contraction if

a) $[\lambda_1, \mu_1], \dots, [\lambda_l, \mu_l]$ are disjoint intervals.

b) $\mu_l < 0$

c) The union of $[\lambda_i, \mu_i]$ covers the spectrum of the linear extension F. A (λ, μ)-contraction F has *a narrow band spectrum* if

$$\mu_i + \mu_l < \lambda_i$$

for $i = 1, \dots, l$

Sub-resonance relation

Let $k_1, \dots, k_l$ be integer nonnegative numbers. We call $(i; k_1, \dots, k_l)$ *a sub–resonance relation* if the following inequality is true

$$\lambda_i \le \sum_j k_j \mu_j$$

We call a monomial $t^{\mathbf{k}}$ for jth coordinate function P_i *a sub–resonance monomial* if $(i;\mathbf{k})$ is a sub–resonance relation. Let us denote by $P_{\lambda,\mu}$ the set of polynomial maps of the sub–resonance type with invertible derivative at the origin and by $G_{\lambda,\mu}$ the group generated by $P_{\lambda,\mu}$.

Lemma

The group $G_{\lambda,\mu}$ is finite–dimensional. More specifically, there exists a number $K = K(\lambda,\mu)$ such that all elements on the group $G_{\lambda,\mu}$ are polynomial maps of degree at most K. Proof: $P_{\lambda,\lambda}$ is always a group.Define $\lambda'_n = \lambda_n, \lambda'_i = \lambda_i \frac{\lambda_n}{\mu_n} \dots \frac{\lambda_{i+1}}{\mu_{i+1}}$.If

$$\lambda_i \le \sum_{i<j} k_j \mu_j$$

Then

$$\lambda_i \frac{\lambda_n}{\mu_n} \dots \frac{\lambda_{i+1}}{\mu_{i+1}} \le \sum_{i<j} k_j \mu_j \frac{\lambda_n}{\mu_n} \dots \frac{\lambda_j}{\mu_j} =$$

$$\sum_{i<j} k_j \lambda_j \frac{\lambda_n}{\mu_n} \dots \frac{\lambda_{j+1}}{\mu_{j+1}} = \sum_{i<j} k_j \lambda'_j$$

Normal form theorem

Suppose extension $\mathcal{F}$ is a contraction and the linear extension $D\mathcal{F}_0$ has a narrow band spectrum determined by vectors $\lambda = (\lambda_1, \dots, \lambda_l)$ and $\mu = (\mu_1, \dots, \mu_l)$. Then there exist a polynomial extension with only subresonance terms smoothly conjugated to $\mathcal{F}$.

Formal change of coordinates

Assume $\mathcal{F}$=$(f(x), F(x,t))$. We look for a change of coordinates in the form $(x, H_x(t)$. We want $P(x,y) = H_{f(x)} \circ F_x(t) \circ H_x^{-1}(t)$ Like before we solve it term by term. We get the following equation for monomial $C_x(t_1, \dots, t_l)$ of degree k:

$$\Phi_i(x) C_x(t_1, \dots, t_l) - C_{f(x)}(\Phi_1(x)t_1, \dots, \Phi_l(x)t_l) = R_x(t)$$

It is a cohomological equation. In non subresonance cas it could be solved using "telescoping sum":

$$\sum_{n=0}^{\infty} \Phi_i^{-1}(x) \dots \Phi_i^{-1}(f^n(x)) R_{f^n(x)}(\Phi(f^{n-1}(x)) \dots \Phi(f(x))(\Phi(x))(t))$$