Analysis C
Functional Analysis — Sample Exam Problems

1. If X is a subset of a Hilbert space H, then let

$$X^\perp = \{ v \in H \mid \langle v, x \rangle = 0 \ \forall x \in X \}.$$

Show that if X is any subset of H, then $(X^\perp)^\perp$ is the smallest closed subspace of H that contains X.

2. Show that a non-empty, closed and convex subset of a Hilbert space has a unique element of minimal norm.

3. Show that every orthonormal sequence in a Hilbert space converges weakly to zero.

4. Let $\{v_1, \ldots, v_n\}$ be an orthonormal set in a Hilbert space H. Show that if $v \in H$, then the quantity

$$\|v - \sum_{k=1}^{n} c_k v_k\|$$

is minimized when and only when $c_k = \langle v_k, v \rangle$ for all k.

5. Let T be a bounded operator on a Hilbert space. Show that T is one-to-one if and only if the adjoint operator T^* has dense range.

6. Let $f : \mathbb{R} \mapsto \mathbb{R}$ be a smooth function with compact support. Prove that

$$\int_{-\infty}^{\infty} |f(x)|^2 \, dx \ \leq \ 2 \left[\int_{\infty}^{\infty} |xf(x)|^2 \, dx \right]^{\frac{1}{2}} \left[\int_{\infty}^{\infty} |f'(x)|^2 \, dx \right]^{\frac{1}{2}}.$$

7. Suppose that $k \in L^2([0,1] \times [0,1])$ and that K is the Hilbert-Schmidt operator on $L^2[0,1]$ that is defined by the formula

$$Kf(x) = \int_0^1 k(x,y) f(y) \, dy.$$

show that if $k(x, y) = \overline{k(y, x)}$ almost everywhere, and if $\{\lambda_n\}$ is the list of eigenvalues of K (including multiplicities), then

$$\sum \lambda_n^2 = \int_0^1 \int_0^1 |k(x, y)|^2 \, dx dy.$$

8. Show that the formula

$$(Kf)(x) = \int_0^x f(y) \, dy$$

determines a compact linear operator K on $L^2[0, 1]$. Find all the eigenvalues of K.

9. Find the maximum and the maximizer of,

$$\int_0^T e^{-(T-t)} x(t) dt,$$

on the unit ball of $L^2[0, T]$.

10. Let H be a Hilbert space, and let $\mathbf{v}_1, \mathbf{v}_2, \ldots$ be a sequence of mutually orthogonal, non-zero vectors. Assume that the set of all finite linear combinations

$$\sum_{i=1}^{N} c_i \mathbf{v}_i$$

is dense on H. Prove that, for every vector $\mathbf{w} \in H$, the series

$$\sum_{i=1}^{\infty} \frac{\langle \mathbf{w}, \mathbf{v}_i \rangle}{\langle \mathbf{v}_i, \mathbf{v}_i \rangle} \mathbf{v}_i$$

converges to $\mathbf{w}$

11. Let $\{\mathbf{e}_1, \mathbf{e}_2, \ldots\}$ be an orthonormal set in a Hilbert space H. Show that the operator $\Lambda : H \mapsto H$ defined as

$$\Lambda \mathbf{v} \; = \; \sum_{k=1}^{\infty} \frac{\langle \mathbf{v}, \mathbf{e}_k \rangle}{k^2} \, \mathbf{e}_k$$

is compact.

12. Let X be the space of all polynomials (of any degree) $f : [0,1] \mapsto \mathbb{R}$, with norm

$$\|f\| \;=\; \max_{x\in[0,1]} |f(x)|\,.$$

Decide whether X is (i) a normed space, and (ii) a Banach space. Check carefully all required properties.

13. Write the definition of the *order* of a distribution T on the real line $\mathbb{R}$.
If $T\phi \geq 0$ for every test function $\phi \geq 0$, show that T must have order zero.

14. Let $\{u_n\}_{n=1}^{+\infty}$ and $\{v_n\}_{n=1}^{+\infty}$ be two orthonormal sets in a Hilbert space H. Assume that

$$\sum_{n=1}^{+\infty} \|u_n - v_n\| < 1.$$

Show that if one set is complete, the other set is also complete.

15. On the Hilbert space $L^2([0,\infty))$, consider the linear operator $\Lambda u(x) = 2u(x+1)$.
(i) Determine $Ker(\Lambda)$ and $Range(\Lambda)$. Is Λ a compact operator ?
(ii) Determine the adjoint operator Λ^*.
(iii) Compute the operator norm $\|\Lambda\|$.

16. Let X be an infinite dimensional Banach space and let $(x_n)_{n\geq 1}$ be a sequence of linearly independent vectors. Define the subspaces $V_n = span\{x_1,\ldots,x_n\}$. Prove that the union

$$V = \bigcup_{n\geq 1} V_n$$

is a subspace of X which is not closed.

17. Let ϕ be a non-zero vector in a Banach space X. Call $U = span\{\phi\} = \{\lambda\phi\,;\;\lambda \in \mathbb{R}\}$.

Prove that there exists a closed subspace $V \subset X$ such that $X = U \oplus V$. Namely, every element $x \in X$ can be written uniquely as a sum

$$x \;=\; u + v \qquad \text{with} \qquad u \in U\,,\;\; v \in V\,.$$

Moreover, show that the projections $x \mapsto u = \pi_U x$ and $x \mapsto v = \pi_V x$ are continuous linear operators.