

Algebra C

Numerical Linear Algebra — Sample Exam Problems

Notation.

Denote by V a finite-dimensional Hilbert space with inner product (\cdot, \cdot) and corresponding norm $\|\cdot\|$. The abbreviation SPD is used for *symmetric positive definite*. If a linear operator $A : V \mapsto V$ is SPD, then $(\cdot, \cdot)_A := (Ax, y)$ is an inner product and the corresponding norm is denoted by $\|\cdot\|_A$. For a square matrix A , $\lambda_j(A)$, $1 \leq j \leq n$ denote the eigenvalues of A . A generic linear iterative method towards the solution of $Ax = b$, $A \in \mathbb{R}^{n \times n}$ is as follows: Given initial guess x_0 , for $k = 0, 1, \dots$ define x_{k+1} as $x_{k+1} = x_k + B(b - Ax_k)$. Different choices of B result in Jacobi, Gauss-Seidel, SOR, Richardson, etc methods.

1. Let $A \in \mathbb{R}^{n \times n}$, and let $A_k = \{a_{ij}\}_{i,j=1}^k$, $k = 1, 2, \dots, n$.

a) Prove that if all of the A_k are nonsingular, then there exists unique decomposition $A = LU$, where L is a *unit* lower triangular matrix and U is an upper triangular matrix.

b) Prove or disprove: if A is nonsingular (but nothing is known about the invertibility of A_k), then there exists a permutation P , and unique pair consisting of a *unit* lower triangular matrix L and an upper triangular matrix U such that $PA = LU$.

2. Consider the conjugate gradient method for the minimization of

$$F(u) := \frac{1}{2}(Au, u) - (b, u),$$

where $A \in \mathbb{R}^{n \times n}$ is SPD. Thus starting with $u_0 = 0$, $r_0 = b$ and $p_0 = r_0$ the successive approximations to the minimizer are computed by

$$\begin{aligned} u_{k+1} &= u_k + \alpha_k p_k, & r_{k+1} &= r_k - \alpha_k A p_k; \\ p_{k+1} &= r_{k+1} - \beta_k p_k, \end{aligned}$$

where $\alpha_k = (r_k, p_k) / \|p_k\|_A^2$ and $\beta_k = -(r_{k+1}, p_k)_A / \|p_k\|_A^2$.

a) Show that for $k = 0, 1, 2, \dots$ the following relations are true:

$$\text{span}\{p_0, p_1, \dots, p_k\} = \text{span}\{r_0, r_1, \dots, r_k\} = \text{span}\{r_0, Ar_0, \dots, A^k r_0\}.$$

b) Show that for some $m \leq n$, $r_m = 0$.

c) Prove that if u_* is the unique minimizer of $F(\cdot)$ then

$$(1) \quad \|u_* - u_k\|_A = \inf_{P \in \mathcal{P}_k; P(0)=1} \|P(A)(u_* - u_0)\|_A,$$

where \mathcal{P}_k is the space of polynomials of real variable of degree $\leq k$.

d) Bound the right side of (1) in terms of the condition number of A .

3. Let $A : V \mapsto V$ be a non-singular linear transformation. Let u be the solution to $Au = f$, and \hat{u} be approximation to it, satisfying $A\hat{u} = \hat{f}$. Show that

$$\frac{\|u - \hat{u}\|_V}{\|u\|} \leq \|A\| \|A^{-1}\| \frac{\|f - \hat{f}\|}{\|f\|}$$

4. Let $A \in \mathbb{R}^{n \times n}$ be SPD.

a) Prove that the SOR (with parameter $0 < \omega < 2$) iterative methods are convergent.

b) Take $n = 3$ and define $A = (I + \sigma vv^T)$, $\sigma > 0$ (A is then SPD). Find a vector $v \in \mathbb{R}^3$, σ , and initial guess x_0 for which Jacobi iterative method will result in a bounded but not convergent sequence of iterates.

5. Let $A : V \mapsto V$ be a linear transformation. Let $W \subseteq V$ be any non-trivial subspace of V . Assume that for every $x \in W$, $x \neq 0$, there exists $y \in W$ such that

$$(Ax, y) \geq \gamma \|x\| \|y\|,$$

where $\gamma > 0$ is independent of x and W .

a) Prove that A is invertible on W , that is, for any $g \in V$ there exists unique $x_W \in W$ such that

$$(2) \quad (Ax_W, y_W) = (g, y_W), \quad \text{for all } y_W \in W.$$

b) Prove that if x_V and x_U are the solutions to (2) with $W = V$ and $W = U \subseteq V$ respectively, then the following estimate holds:

$$\|x_V - x_W\| \leq K \inf_{y \in W} \|x_V - y\|,$$

Here K is a constant which depends only on $\|A\|$ and γ .

6. Let $A \in \mathbb{R}^{n \times n}$ be irreducible. Assume also that the following inequalities hold: for all $1 \leq i \leq n$, and $1 \leq j \leq n$, $a_{ii} > 0$, $a_{ij} \leq 0$, $i \neq j$,

$$a_{ii} + \sum_{i=1, i \neq j}^n a_{ij} \geq 0,$$

with strict inequality for at least one i .

a) Prove that if $f \in \mathbb{R}^n$ is such that $f_j \geq 0$ (resp. ≤ 0), for all $1 \leq j \leq n$, then the solution u to $Au = f$ satisfies $u_j \geq 0$ (resp. $(\leq)0$), for all $1 \leq j \leq n$.

b) Prove that A is invertible.

c) Prove that both Gauss-Seidel and Jacobi iterative methods are convergent for this type of matrices.

7. Suppose the entries of $A(t) \in \mathbb{R}^{n \times n}$ are continuously differentiable functions of the scalar t . Assume that $A(0)$ is non-singular and all its principal sub-matrices are non-singular. Show that for sufficiently small $|t|$, there exists an LU-factorization of A and the factors $L(t)$ and $U(t)$ are also continuously differentiable.

8. If A is real $m \times n$ matrix and U, V are orthogonal matrices, such that $U^T A V = \text{diag}(\sigma_1, \dots, \sigma_p)$, $p = \min\{m, n\}$ (singular value decomposition of A), then

$$\min_{\text{rank}(B)=k} \|A - B\|_{\ell_2} = \|A - A_k\|_{\ell_2} = \sigma_{k+1}, \quad \text{for } k < r = \text{rank}(A)$$

Here $A_k = \sum_{i=1}^k \sigma_i u_i v_i^T$, and u_i and v_i are the columns of U and V , respectively.

9. Let $A \in \mathbb{R}^{n \times n}$ and $B \in \mathbb{R}^{n \times n}$. Find $\left. \frac{d}{dt} \det(A + tB) \right|_{t=0}$.

10. Let $A \in \mathbb{R}^{m \times n}$, $m \geq n$, have rank n , and let $b \in \mathbb{R}^m$.

a) Show that there exists a unique $x \in \mathbb{R}^n$ minimizing $\|Ax - b\|_{\ell_2}^2$

b) Show that the matrix $A^T A$ is invertible and $x = (A^T A)^{-1} A^T b$.

c) Given the QR factorization of A , write down the solution of the least squares problem in terms of the components of the factorization.

11. Let $A = (a_{ij})$ be an SPD matrix of order n . After one step of Gaussian elimination A is converted to a matrix of the form $\begin{pmatrix} a_{11} & a^T \\ 0 & \tilde{A} \end{pmatrix}$. Show that the $(n-1) \times (n-1)$ matrix \tilde{A} is SPD.

12. Let $P \in \mathbb{R}^{n \times m}$ and $Q \in \mathbb{R}^{m \times n}$ and $A \in \mathbb{R}^{n \times n}$. Assuming that A and $(I + QA^{-1}P)$ are invertible, prove that $(A + PQ)$ is invertible and

$$(A + PQ)^{-1} = A^{-1} - A^{-1}P(I + QA^{-1}P)^{-1}QA^{-1}.$$

13. Let $A \in \mathbb{C}^{n \times n}$, and define $\rho(A) = \max_{1 \leq j \leq n} |\lambda_j(A)|$, $\alpha(A) = \max_{1 \leq j \leq n} \operatorname{Re} \lambda_j(A)$. Prove that:

a) There exists a unitary matrix Q and an upper triangular matrix T , such that $A = QTQ^*$.

b) $\lim_{k \rightarrow \infty} \|A^k\| = 0$, iff $\rho(A) < 1$.

c) $\lim_{t \rightarrow \infty} \|\exp(tA)\| = 0$, iff $\alpha(A) < 0$.

14. Let $x \in \mathbb{R}^n$, and $y \in \mathbb{R}^n$ be such that $\|x\|_{\ell_2} = \|y\|_{\ell_2}$.

a) Prove that there exists a symmetric orthogonal matrix H such that $Hx = y$.

b) Show that for a given $A \in \mathbb{R}^{m \times n}$ one can construct symmetric orthogonal matrices (reflections) $H^{(k)}$, $k = 1, \dots, \ell$ such that

$$A^{(\ell+1)} = H^{(\ell)}H^{(\ell-1)} \dots H^{(1)}A, \ell \leq \min(m-1, n)$$

has row echelon structure, that is, $A_{kj}^{(\ell+1)} = 0$, for $j < k$.

15. Let $A \in \mathbb{R}^{n \times n}$ be a strictly diagonally dominant square matrix and C_L be the lower triangle of A (including the diagonal). Prove that $\rho(I - C_L^{-1}A) \leq \max_{1 \leq k \leq n} \delta_k$

where

$$\delta_k = \left\{ \sum_{j=k+1}^n |a_{kj}| \right\} / \left\{ |a_{kk}| - \sum_{j=1}^{k-1} |a_{kj}| \right\}.$$

16. Let A and B be symmetric real matrices.

a) Prove that all tridiagonal symmetric Toeplitz matrices commute with each other.

b) Let A and B be SPD matrices. Prove that all the eigenvalues of AB are positive.

17. Consider the LDL^t factorization of an $n \times n$ tridiagonal positive definite Toeplitz matrix A :

$$A = LDL^t, \quad D = \begin{pmatrix} d_1 & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & d_n \end{pmatrix} \quad L = \begin{pmatrix} 1 & 0 & \dots & 0 & 0 \\ l_{21} & 1 & \dots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \dots & l_{n,n-1} & 1 \end{pmatrix}.$$

Show that d_n and $l_{n,n-1}$ converge as $n \rightarrow \infty$.