Analysis C
Functional Analysis — Sample Exam Problems

1. If X is a subset of a Hilbert space H, then let

$$X^{\perp} = \{ v \in H \mid \langle v, x \rangle = 0 \ \forall x \in X \}.$$

Show that if X is any subset of H, then $X^{\perp\perp}$ is the smallest closed subspace of H that contains X.

2. Show that a non-empty, closed and convex subset of a Hilbert space has a unique element of minimal norm.

3. Show that every orthonormal sequence in a Hilbert space converges weakly to zero.

4. Let $\{v_1, \ldots, v_n\}$ be an orthonormal set in a Hilbert space H. Show that if $v \in H$, then the quantity

$$\|v - \sum_{k=1}^{n} c_k v_k\|$$

is minimized when and only when $c_k = \langle v_k, v \rangle$ for all k.

5. Let T be a bounded operator on a Hilbert space. Show that T is one-to-one if and only if the adjoint operator T^* has dense range.

6. Let f be a smooth and compactly supported function on $\mathbb{R}$. Show that

$$\int_{-\infty}^{\infty} |f(x)|^2 \, dx \le 2 \left[\int_{\infty}^{\infty} |xf(x)|^2 \, dx \right]^{\frac{1}{2}} \left[\int_{\infty}^{\infty} |f'(x)|^2 \, dx \right]^{\frac{1}{2}}.$$

7. Suppose that $k \in L^2([0,1] \times [0,1])$ and that K is the Hilbert-Schmidt operator on $L^2[0,1]$ that is defined by the formula

$$Kf(x) = \int_0^1 k(x,y) f(y) \, dy.$$

show that if $k(x, y) = \overline{k(y, x)}$ almost everywhere, and if $\{\lambda_n\}$ is the list of eigenvalues of K (including multiplicities), then

$$\sum \lambda_n^2 = \int_0^1 \int_0^1 |k(x, y)|^2 \, dx dy.$$

8. Show that the formula

$$Vf(x) = \int_0^x f(y) \, dy$$

determines a compact operator on $L^2[0, 1]$. Find all the eigenvalues of V.

9. Show that the formula

$$\Lambda(f) = \lim_{\varepsilon \to 0} \int_{|x| \geq \varepsilon} \frac{f(x)}{x} \, dx$$

defines a tempered distribution on $\mathbb{R}$.

10. Prove that the Fourier transform is a homeomorphism from the Schwartz space of $\mathbb{R}$ onto itself.

11. Show that the function $f(x) = e^x \cos(e^x)$ is a tempered distribution on $\mathbb{R}$.

12. Give an example of a distribution Λ on $\mathbb{R}$ and a smooth, compactly supported function f on $\mathbb{R}$ such that f vanishes on the support of Λ, yet $\Lambda(f) \neq 0$.

13. Let Λ be a distribution on $\mathbb{R}$ that has order N and support $\{0\}$. Show that if f is any smooth, compactly supported function and $g(x) = x^{N+1} f(x)$, then $\Lambda(g) = 0$.

14. Characterize (with proof) all distributions Λ on $\mathbb{R}$ for which $\frac{d}{dx}\Lambda = 0$.

15. Let f be a smooth, periodic function on $\mathbb{R}$. Show that f is a tempered distribution and $\mathbb{R}$, and describe (with proof) the relationship between the Fourier transform of f and its Fourier series.

16. Show that the function $f(x,y) = (x+iy)^{-1}$ is a tempered distribution on $\mathbb{R}^2$ and compute its Fourier transform. Use your answer to determine a fundamental solution for the operator $\frac{d}{dx} + \sqrt{-1}\frac{d}{dy}$ on $\mathbb{R}^2$.

17. Show that the function

$$k(x,y) = \begin{cases} \frac{1}{\sqrt{4\pi x}} e^{-\frac{y^2}{4x}} & x > 0 \\ 0 & x \leq 0 \end{cases}$$

is a fundamental solution for the heat operator $\frac{d}{dx} - \frac{d^2}{dy^2}$.