Analysis B
Complex Analysis — Sample Exam Problems

1. Show that the function $f(z) = z\,\Re z$ is complex differentiable at $z = 0$, but it is not holomorphic there.

2. Write the Cauchy-Riemann equations in polar coordinates (r, θ). Then, show that the logarithm, defined by $\log(z) = \log r + i\,\theta$, $z = r\,e^{i\,\theta}$, is holomorphic in the region $r > 0$, $-\pi < \theta < +\pi$.

3. Prove that the power series:

$$\sum_{n=0}^{+\infty} \frac{z^n}{n}$$

is uniformly convergent on any arc $\gamma = \{z = e^{i\theta},\ \alpha \le \theta \le \beta\}$ on the unit circle that does not contain the point $z = 1$. (Hint: use summation by parts formula for series.)

4. Let a, b be given complex numbers, $|a| < |b|$. Show directly that if $|a| < r < |b|$, then

$$\int_{\mathcal{C}} \frac{1}{(z-a)(z-b)}\,dz = \frac{2\pi i}{a-b},$$

where $\mathcal{C}$ is the circle with center the origin and radius r, oriented counterclockwise.

5. Calculate the following *Fresnel integrals*:

$$\int_0^{+\infty} \cos(x^2)\,dx, \qquad \int_0^{+\infty} \sin(x^2)\,dx.$$

6. Let Ω be an open set in $\mathbb{C}$ and fix $w \in \Omega$. Prove that if f is holomorphic in $\Omega \setminus \{w\}$ and bounded near w, then

$$\int_T f(z)\,dz = 0,$$

for any triangle T, the interior of which lies in Ω. Deduce that f can be extended to a holomorphic function in Ω.

7. Let f be holomorphic in the unit disk $\mathbb{D}$. Prove that for any $a \in \mathbb{D}$,

$$f(a) = \frac{1}{2\pi i} \int_{\mathcal{C}} \frac{1 - a\,\bar{a}}{(z-a)(1-z\,\bar{a})} f(z)\, dz,$$

where $\mathcal{C}$ is the circle with center the origin and radius r, $|a| < r < 1$, oriented counterclockwise.

8. Show that if the real part of an entire function is bounded then f is constant.

9. Classify all singularities of the following function

$$f(z) = \frac{e^z - 1}{z(z-1)}.$$

10. Prove that an entire function $f(z)$ is a polynomial if and only if the only singularity for f on the *extended* complex plane $\mathbb{C} \cup \{\infty\}$ is a pole at $z = \infty$.

11. Evaluate the following integrals using the residue formula:

1. $\displaystyle\int_0^{2\pi} \frac{d\theta}{1 - 2a\,\cos\theta + a^2}, \quad |a| < 1;$
2. $\displaystyle\int_{-\infty}^{+\infty} \frac{x \sin x}{x^2 + a^2}\, dx, \quad a \geq 0;$
3. $\displaystyle\int_{-\infty}^{+\infty} \frac{e^{a\,x}}{1 + e^x}\, dx, \quad 0 < a < 1.$

12. Let f be holomorphic on a neighborhood of the closed unit disk $\bar{\mathbb{D}}$, such that $|f(z)| < 1$ if $|z| = 1$. Determine how many fixed points f has in the disk.

13. Prove that all injective, entire functions take the form $f(z) = az + b$, $a, b \in \mathbb{C}$, $a \neq 0$. (Hint: apply Casorati-Weierstrass Theorem to the function $f(1/z)$.)

14. Let $P(z)$ be a polynomial of degree n with distinct real roots $0 \leq z_1 < z_2 < z_3 < \ldots < z_n$. Find all possible values of

$$I(r) = \int_{\{z \in \mathbb{C} \,\mid\, |z| = r\}} \frac{z\,P'(z)}{P(z)}\, dz, \quad 0 < r < \infty,$$

where $r \neq z_j$, $j = 1, \ldots, n$.

15. Determine all branch points of the following functions:

$$f(z) = \log(z^2 - 1), \qquad g(z) = \sqrt{z^2 - 1}.$$

Then cut the complex plane appropriately to determine a single-valued branch of f and g.

16. Evaluate the following integrals using countour integration:

1. $\displaystyle\int_0^{+\infty} \log \sin \theta \, d\theta$;
2. $\displaystyle\int_0^{+\infty} \frac{\log(1 + x^2)}{x^{1+\alpha}} \, dx, \quad 0 < \alpha < 2.$ (Hint: integrate by parts.)

17. Find a one-to-one conformal map $F(z) = w$ of the region common to the disks $|z| < 1$ and $|z - 1| < 1$ onto the exterior or interior of the unit circle $|w| = 1$. Write your answer as a composition of elementary maps.

18. Prove that all conformal maps from the upper half plane $\mathbb{H}$ to the unit disk $\mathbb{D}$ are of the form:

$$F(z) = e^{i\theta} \frac{z - \beta}{z - \bar{\beta}}, \quad \theta \in \mathbb{R}, \ \beta \in \mathbb{H}.$$