

Math 558, Foundations of Mathematics

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Some Homework Problems

1. Show that the function $f = \lambda n [nth \text{ digit of } \sqrt{2}]$ is primitive recursive.

Solution: Using the bounded least number operator, define a primitive recursive function

$$\begin{aligned} g(n) &= \lfloor \sqrt{2} \cdot 10^n \rfloor \\ &= \text{least } x \text{ such that } \left(\frac{x+1}{10^n} \right)^2 > 2 \\ &\quad \text{or, equivalently, } (x+1)^2 > 2 \cdot 10^{2n} \\ &= \text{least } x < 2 \cdot 10^n \text{ such that } (x+1)^2 > 2 \cdot 10^{2n}; \end{aligned}$$

thus $g(0) = 1$, $g(1) = 14$, $g(2) = 141$, $g(3) = 1414$, $g(4) = 14142$, etc.
It follows that

$$f(n) = \text{Remainder}(g(n), 10)$$

is also primitive recursive.

2. Exercise on the Ackermann function.

(a) Show that each F_n is primitive recursive.

(b) i. $F_n(x) > 0$.

ii. $F_n(x+1) > F_n(x)$, *i.e.* F_n is monotone.

iii. $F_n(x) > x$.

iv. $F_{n+1} \geq F_n(x+1)$.

- (c) Every primitive recursive function is dominated by F_n for some n .
- (d) $\lambda x[F_x(x)]$ is not primitive recursive.

Solution. Recall the definition

$$\begin{aligned} F_0(x) &= x + 1 \\ F_{n+1}(x) &= \underbrace{F_n \cdots F_n}_{x+1}(1). \end{aligned}$$

- (a) By induction on n . Obviously F_0 is primitive recursive. The recursion equations

$$\begin{aligned} F_{n+1}(0) &= F_n(1) \\ F_{n+1}(x + 1) &= F_n(F_{n+1}(x)) \end{aligned}$$

show that if F_n is primitive recursive then so is F_{n+1} .

- (b) $F_n(x) > 0$ is easily proved by induction on n . Prove F_n monotone and $F_n(x) > x$ simultaneously by induction on n : $F_n(0) > 0$ and monotonicity imply $F_n(x) > x$ for all x , hence $F_{n+1}(x + 1) = F_n(F_{n+1}(x)) > F_{n+1}(x)$. Prove $F_{n+1}(x) \geq F_n(x + 1)$ by induction on x : For $x = 0$ we have $F_{n+1}(0) = F_n(1)$, and for $x + 1$ we have $F_{n+1}(x + 1) = F_n(F_{n+1}(x)) \geq F_n(F_n(x + 1))$ by inductive hypothesis, and $F_n(F_n(x + 1)) \geq F_n(x + 2)$ since $F_n(x + 1) \geq x + 2$ and F_n is monotone.
- (c) Recall that $f(x_1, \dots, x_k)$ is said to be dominated by F_n if

$$f(x_1, \dots, x_k) \leq F_n(\max(x_1, \dots, x_k))$$

for all x_1, \dots, x_k . Clearly the initial functions are dominated by F_0 . For composition, if $f = h(g_1, \dots, g_m)$, let n be sufficiently large so that F_{n+1} dominates g_1, \dots, g_m and F_n dominates h . Then an easy computation shows that F_{n+1} dominates f . For primitive recursion, if f is obtained by primitive recursion from g and h , let n be sufficiently large so that F_{n+1} dominates g and F_n dominates h . We claim that

$$f(y, x_1, \dots, x_k) \leq F_{n+1}(y + \max(x_1, \dots, x_k))$$

for all x_1, \dots, x_k, y . This is easily proved by induction on y . Note also that

$$F_{n+2}(z) = F_{n+1} \underbrace{F_{n+1} \cdots F_{n+1}}_z(1) \geq F_{n+1}(2z + 1)$$

for all z , since $F_{n+1}(w) \geq F_1(w) = w + 2$ for all w . Thus we have

$$\begin{aligned} f(y, x_1, \dots, x_k) &\leq F_{n+1}(y + \max(x_1, \dots, x_k)) \\ &\leq F_{n+1}(2 \max(y, x_1, \dots, x_k)) \\ &\leq F_{n+2}(\max(y, x_1, \dots, x_k)) \end{aligned}$$

and this completes the proof.

(d) If $\lambda x[F_x(x)]$ were primitive recursive, then $\lambda x[F_x(x) + 1]$ would be primitive recursive, hence dominated by F_n for some n , in particular $F_n(n) + 1 \leq F_n(n)$, a contradiction.

3. Show that every computable function can be computed by a register machine with 4 registers.
4. Show that a total recursive function $f(x_1, \dots, x_k)$ is primitive recursive if and only if there exists an index e of f such that

$$\lambda x_1 \cdots x_k [\text{Stop}(e, x_1, \dots, x_k)]$$

is dominated by some primitive recursive function.

5. Show that

$$T = \{x \mid \varphi_x^{(1)} \text{ is total}\}$$

and

$$E = \{x \mid \varphi_x^{(1)} \text{ is the empty function}\}$$

are nonrecursive.

Solution: By the Enumeration and Parametrization theorems, we can find a primitive recursive function f such that

$$\varphi_{f(x)}^{(1)}(y) \simeq \varphi_x^{(1)}(x)$$

for all x and y . Then $x \in K$ implies $f(x) \in T$, while $x \notin K$ implies $f(x) \in E$. Thus f reduces K to T and to the complement of E .

6. Prove Rice's Theorem: If \mathcal{C} is any nontrivial class of 1-place partial recursive functions, then the index set $I_{\mathcal{C}} = \{x \mid \varphi_x^{(1)} \in \mathcal{C}\}$ is nonrecursive.

Solution: Let x_0 be an index of the empty function, and let x_1 be an index such that $\varphi_{x_1}^{(1)} \in \mathcal{C}$ if and only if $\varphi_{x_0}^{(1)} \notin \mathcal{C}$. By the Enumeration and Parametrization theorems, we can find a primitive recursive function f such that

$$\varphi_{f(x)}^{(1)}(y) \simeq \begin{cases} \varphi_{x_1}^{(1)}(y) & \text{if } \varphi_x^{(1)}(x) \text{ is defined} \\ \text{undefined} & \text{otherwise} \end{cases}$$

for all x and y . Thus $x \in K$ implies $\varphi_{f(x)}^{(1)} = \varphi_{x_1}^{(1)}$, while $x \notin K$ implies $\varphi_{f(x)}^{(1)} = \varphi_{x_0}^{(1)}$. Thus f reduces K either to $I_{\mathcal{C}}$ (if $\varphi_{x_1}^{(1)} \in \mathcal{C}$) or to the complement of $I_{\mathcal{C}}$ (if $\varphi_{x_1}^{(1)} \notin \mathcal{C}$). In either case it follows that $I_{\mathcal{C}}$ is not recursive.

7. Show that a predicate $P \subseteq \mathbf{N}^k$ is Δ_2^0 if and only if χ_P is a limit-recursive function.

Solution:

For simplicity, let x be an abbreviation for x_1, \dots, x_k .

First assume that χ_P is limit-recursive, say

$$\chi_P(x) = \lim_n f(n, x)$$

for all x , where $f(n, x)$ is a recursive function. Then we have

$$P(x) \equiv \exists m \forall n (n > m \rightarrow f(n, x) = 1)$$

and

$$\neg P(x) \equiv \exists m \forall n (n > m \rightarrow f(n, x) = 0)$$

so P is Δ_2^0 .

For the converse, assume that P is Δ_2^0 , say

$$P(x) \equiv \exists y \forall z R_1(x, y, z)$$

and

$$\neg P(x) \equiv \exists y \forall z R_0(x, y, z)$$

where R_1 and R_0 are primitive recursive predicates. Define $g(n, x) =$ the least $y < n$ such that either $\forall z < n R_1(x, y, z)$ or $\forall z < n R_0(x, y, z)$ or both, if such a y exists, and $g(n, x) = n$ otherwise. Thus $g(n, x)$ is a primitive recursive function, and it is easy to see that for all x , $g(x) = \lim_n g(n, x)$ exists, and $g(x) =$ the least y such that $\forall z R_1(x, y, z)$ or $\forall z R_0(x, y, z)$. Now define $h(n, x) = 1$ if $\forall z < n R_1(x, g(n, x), z)$, and $h(n, x) = 0$ otherwise. Thus $h(n, x)$ is again a primitive recursive function, and for all x , $h(x) = \lim_n h(n, x)$ exists. Moreover $P(x)$ implies $h(x) = 1$, and $\neg P(x)$ implies $h(x) = 0$. Thus χ_P is limit-recursive. This completes the proof.

8. Write down a sentence expressing Goldbach's Conjecture: every even number > 2 is the sum of two primes. Write down a formula defining the function λxy [least common multiple of x and y].
9. Prove König's theorem: If $\kappa_i, \lambda_i, i \in I$ are cardinals and $\kappa_i < \lambda_i$ for all $i \in I$, then $\sum_{i \in I} \kappa_i < \prod_{i \in I} \lambda_i$.
10. Prove Zorn's lemma: If \mathcal{S} is a set of sets which is closed under unions of chains, then \mathcal{S} has a maximal element.
11. Show that $2^{\aleph_0} \neq \aleph_\omega$. (Hint: Use König's theorem.) Generalize.
12. Prove that the real number system (defined in terms of Cauchy sequences) is complete, *i.e.*, every nonempty bounded set of real numbers has a least upper bound.
13. Prove the Generalized Löwenheim-Skolem theorem: Let κ be an infinite cardinal, let (A, E) be a relational structure with $|A| \geq \kappa$, and let $X \subseteq A$ be such that $|X| \leq \kappa$. Then there exists an elementary substructure (A', E') of (A, E) such that $X \subseteq A'$ and $|A'| = \kappa$.
14. Recall that $L_\omega = R_\omega = \{\text{hereditarily finite sets}\}$. For $X \subseteq \omega$, show that $X \in L_{\omega+1}$ if and only if X is arithmetically definable, *i.e.* belongs to the arithmetical hierarchy, *i.e.* $X \in \Sigma_k^0$ for some $k \in \mathbf{N}$.