

Math 558, Foundations of Math I, Spring 1995

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Homework Problems

1. Exhibit construction sequences showing that $\lambda xy[xy]$ and $\lambda x[2^x]$ are primitive recursive.
2. Show that $\lambda mnk[k\text{th digit of } F(m)/F(n)]$ is primitive recursive, where $F(n) = n\text{th Fibonacci number}$. You may take for granted the material of Section 1 of the notes.

Solution. The following are primitive recursive:

$$\begin{aligned}\ell(n) &= \text{number of digits in } n \\ &= \text{least } \ell \text{ such that } 10^\ell > n\end{aligned}$$

assuming $n > 0$;

$$\begin{aligned}d_0(n, k) &= k^{\text{th}} \text{ digit of } n \\ &= \text{Rem}(\text{Quot}(n, 10^{\ell(n)-k}), 10)\end{aligned}$$

assuming $n > 0$ and has at least k digits;

$$d_1(m, n, k) = d_0(\text{Quot}(10^j m, n), k)$$

where j is sufficiently large so that $\text{Quot}(10^j m, n)$ has at least k digits, say $j = \text{least } j \text{ such that } 10^j m > 10^k n$. Note that $d_1(m, n, k)$ is the k^{th} digit of the rational number m/n provided $m > n > 0$. Next define

$$d_2(m, n, k) = \begin{cases} d_1(m, n, k) & \text{if } m > n \\ d_1(m + n, n, k + 1) & \text{if } m < n \\ 1 & \text{if } m = n \text{ and } k = 1 \\ 0 & \text{if } m = n \text{ and } k > 1 \end{cases}$$

Thus $d_2(m, n, k)$ is the k^{th} digit of the rational number m/n for all $m, n > 0$. Finally the desired function is $d_2(F(m), F(n), k)$.

3. Exhibit a register machine program to compute $\lambda xy[xy]$.

Solution. See Chapter 1.

4. Exercise on the Ackermann function.

1. Show that each F_n is primitive recursive.
2. (a) $F_n(x) > 0$.
 (b) $F_n(x + 1) > F_n(x)$, i.e. F_n is monotone.
 (c) $F_n(x) > x$.
 (d) $F_{n+1} \geq F_n(x + 1)$.
3. Every primitive recursive function is dominated by F_n for some n .
4. $\lambda x[F_x(x)]$ is not primitive recursive.

Solution. Recall the definition

$$\begin{aligned} F_0(x) &= x + 1 \\ F_{n+1}(x) &= \underbrace{F_n \cdots F_n}_{x+1}(1). \end{aligned}$$

1. By induction on n . Obviously F_0 is primitive recursive. The recursion equations

$$\begin{aligned} F_{n+1}(0) &= F_n(1) \\ F_{n+1}(x + 1) &= F_n(F_{n+1}(x)) \end{aligned}$$

show that if F_n is primitive recursive then so is F_{n+1} .

2. $F_n(x) > 0$ is easily proved by induction on n . Prove F_n monotone and $F_n(x) > x$ simultaneously by induction on n : $F_n(0) > 0$ and monotonicity imply $F_n(x) > x$ for all x , hence $F_{n+1}(x + 1) = F_n(F_{n+1}(x)) > F_{n+1}(x)$. Prove $F_{n+1}(x) \geq F_n(x + 1)$ by induction on x : For $x = 0$ we have $F_{n+1}(0) = F_n(1)$, and for $x + 1$ we have $F_{n+1}(x + 1) = F_n(F_{n+1}(x)) \geq F_n(F_n(x + 1))$ by inductive hypothesis, and $F_n(F_n(x + 1)) \geq F_n(x + 2)$ since $F_n(x + 1) \geq x + 2$ and F_n is monotone.

3. Recall that $f(x_1, \dots, x_k)$ is said to be dominated by F_n if

$$f(x_1, \dots, x_k) \leq F_n(\max(x_1, \dots, x_k))$$

for all x_1, \dots, x_k . Clearly the initial functions are dominated by F_0 . For composition, if $f = h(g_1, \dots, g_m)$, let n be sufficiently large so that F_{n+1} dominates g_1, \dots, g_m and F_n dominates h . Then an easy computation shows that F_{n+1} dominates f . For primitive recursion, if f is obtained by primitive recursion from g and h , let n be sufficiently large so that F_{n+1} dominates g and F_n dominates h . We claim that

$$f(y, x_1, \dots, x_k) \leq F_{n+1}(y + \max(x_1, \dots, x_k))$$

for all x_1, \dots, x_k, y . This is easily proved by induction on y . Note also that

$$F_{n+2}(z) = F_{n+1} \underbrace{F_{n+1} \cdots F_{n+1}}_z(1) \geq F_{n+1}(2z + 1)$$

for all z , since $F_{n+1}(w) \geq F_1(w) = w + 2$ for all w . Thus we have

$$\begin{aligned} f(y, x_1, \dots, x_k) &\leq F_{n+1}(y + \max(x_1, \dots, x_k)) \\ &\leq F_{n+1}(2 \max(y, x_1, \dots, x_k)) \\ &\leq F_{n+2}(\max(y, x_1, \dots, x_k)) \end{aligned}$$

and this completes the proof.

4. If $\lambda x[F_x(x)]$ were primitive recursive, then $\lambda x[F_x(x) + 1]$ would be primitive recursive, hence dominated by F_n for some n , in particular $F_n(n) + 1 \leq F_n(n)$, a contradiction.

5. Show that

$$T = \{x \mid \varphi_x^{(1)} \text{ is total}\}$$

and

$$E = \{x \mid \varphi_x^{(1)} \text{ is the empty function}\}$$

are nonrecursive.

Solution. By the Enumeration and Parametrization theorems, we can find a primitive recursive function f such that

$$\varphi_{f(x)}^{(1)}(y) \simeq \varphi_x^{(1)}(x)$$

for all x and y . Then $x \in K$ implies $f(x) \in T$, while $x \notin K$ implies $f(x) \in E$. Thus f reduces K to T and to the complement of E .

6. Prove Rice's Theorem: If \mathcal{C} is any nontrivial class of 1-place partial recursive functions, then the index set $I_{\mathcal{C}} = \{x \mid \varphi_x^{(1)} \in \mathcal{C}\}$ is nonrecursive.

Solution. Let x_0 be an index of the empty function, and let x_1 be an index such that $\varphi_{x_1}^{(1)} \in \mathcal{C}$ if and only if $\varphi_{x_0}^{(1)} \notin \mathcal{C}$. By the Enumeration and Parametrization theorems, we can find a primitive recursive function f such that

$$\varphi_{f(x)}^{(1)}(y) \simeq \begin{cases} \varphi_{x_1}^{(1)}(y) & \text{if } \varphi_x^{(1)}(x) \text{ is defined} \\ \text{undefined} & \text{otherwise} \end{cases}$$

for all x and y . Thus $x \in K$ implies $\varphi_{f(x)}^{(1)} = \varphi_{x_1}^{(1)}$, while $x \notin K$ implies $\varphi_{f(x)}^{(1)} = \varphi_{x_0}^{(1)}$. Thus f reduces K either to $I_{\mathcal{C}}$ (if $\varphi_{x_1}^{(1)} \in \mathcal{C}$) or to the complement of $I_{\mathcal{C}}$ (if $\varphi_{x_1}^{(1)} \notin \mathcal{C}$). In either case it follows that $I_{\mathcal{C}}$ is not recursive.

7. Write down a sentence expressing Goldbach's Conjecture: every even number > 2 is the sum of two primes. Write down a formula defining the function λxy [least common multiple of x and y].

8. Show that a total recursive function $f(x_1, \dots, x_k)$ is primitive recursive if and only if there exists an index e of f such that $\lambda x_1 \cdots x_k [\text{Stop}(e, x_1, \dots, x_k)]$ is dominated by some primitive recursive function.

9. Show that the definitions of $\sum_{i \in I} \kappa_i$ and $\prod_{i \in I} \kappa_i$ are valid. Specifically, you need to prove the following:

1. Given an indexed set of cardinal numbers $\langle \kappa_i \rangle_{i \in I}$, there exists an indexed set of sets $\langle X_i \rangle_{i \in I}$ such that $\text{card}(X_i) = \kappa_i$ for all $i \in I$ and $X_i \cap X_j = \emptyset$ for all $i, j \in I$ with $i \neq j$.
2. If $\langle X'_i \rangle_{i \in I}$ is another such indexed set of sets, then $\bigcup_{i \in I} X_i \approx \bigcup_{i \in I} X'_i$ and $\prod_{i \in I} X_i \approx \prod_{i \in I} X'_i$.

10. Prove König's theorem: If $\langle \kappa_i \rangle_{i \in I}$ and $\langle \lambda_i \rangle_{i \in I}$ are indexed sets of cardinal numbers with $\kappa_i < \lambda_i$ for all $i \in I$, then $\sum_{i \in I} \kappa_i < \prod_{i \in I} \lambda_i$.

11. Define $\aleph_0 = \text{card}(\mathbb{N})$. Show that $\text{card}(\mathbb{Q}) = \aleph_0$, $\text{card}(\mathbb{R}) = 2^{\aleph_0}$, $\text{card}(\mathbb{R}^{\mathbb{N}}) = 2^{\aleph_0}$, $\text{card}(\mathbb{R}^{\mathbb{R}}) = 2^{2^{\aleph_0}}$.

12. If F is a continuous increasing function from ordinals to ordinals, show that F has arbitrarily large fixed points. (*Increasing* means that $F(\alpha) \geq \alpha$ for all α . *Continuous* means $F(\delta) = \sup\{F(\alpha) \mid \alpha < \delta\}$ for limit ordinals δ . A *fixed point* of F is an ordinal α such that $F(\alpha) = \alpha$.)

13. Show that $2^{\aleph_0} \neq \aleph_\omega$. (Hint: Use König's theorem.)

14. Show that, if there exist at least $n + 1$ inaccessible cardinals, then there is a transitive model of ZFC + "there exist exactly n inaccessible cardinals."