Math 557 (Mathematical Logic I)
November 27, 1991

TAKEHOME FINAL EXAM

If A is a set and $B \subseteq A$, then for any k-ary relation $R \subseteq A^k$ on A, we write $R|B = R \cap B^k$. This is the restriction of R to B. Similarly, if $f : A^k \to A$ is a k-ary operation on A, we write $f|B$ for the restriction of f to B, *i.e.* the unique function $f' : B^k \to A$ such that $f'(b_1, \ldots, b_k) = f(b_1, \ldots, b_k)$ for all $b_1, \ldots, b_k \in B$. We say that B is *closed* under f if $f|B$ is an operation on B, *i.e.* the range of $f|B$ is included in B.

Given a structure

$$\mathcal{A} = (A, R^{\mathcal{A}}, \ldots, f^{\mathcal{A}}, \ldots, c^{\mathcal{A}}, \ldots)\,,$$

a *substructure* of $\mathcal{A}$ is a structure

$$\mathcal{B} = (B, R^{\mathcal{B}}, \ldots, f^{\mathcal{B}}, \ldots, c^{\mathcal{B}}, \ldots)$$

where $B \subseteq A$ and $R^{\mathcal{B}} = R^{\mathcal{A}}|B$, $\ldots$, $f^{\mathcal{B}} = f^{\mathcal{A}}|B$, $\ldots$, $c^{\mathcal{B}} = c^{\mathcal{A}}$, $\ldots$. This implies that B contains the constants $c^{\mathcal{A}}$, $\ldots$, and is closed under the operations $f^{\mathcal{A}}$, $\ldots$. We write $\mathcal{B} \subseteq \mathcal{A}$ to denote that $\mathcal{B}$ is a substructure of $\mathcal{A}$.

Recall that a formula α is Π_k if it is of the form

$$\forall \overline{x}_1 \exists \overline{x}_2 \ldots \overline{x}_k \gamma$$

where γ is quantifier-free. Thus α consists of k alternating blocks of quantifiers, the first block being universal, followed by a quantifier-free matrix. Similarly, a Σ_k formula is one consisting of k alternating blocks of quantifiers, the first block being existential, followed by a quantifier-free matrix.

A *sentence* is a formula with no free variables.

1. Let α be a first order sentence. Show that the following conditions are equivalent. (a) α is logically equivalent to a Π_1 sentence. (b) For any structure $\mathcal{A}$ and substructure $\mathcal{B} \subseteq \mathcal{A}$, if $\mathcal{A} \models \alpha$ then $\mathcal{B} \models \alpha$.

2. Recall that, by the theorem on prenex forms, any formula is logically equivalent to a Π_k formula for some $k \in \mathbb{N}$. (a) Give an example of a Π_1 sentence which is not logically equivalent to any quantifier-free sentence. (b) Give an example of a Π_2 sentence which is not logically equivalent to any Π_1 sentence. (c) Give an example of a Π_3 sentence which is not logically equivalent to any Π_2 sentence. (d) Show that, for any $k \in \mathbb{N}$, there exists a sentence α such that α is not logically equivalent to any Π_k sentence.

Let L be a first-order language, and let Φ be a set of L-sentences. Recall that Φ is said to be *complete* if, for all L-sentences α, either $\Phi \vdash \alpha$ or $\Phi \vdash \neg\alpha$. 3.

Assume that Φ is consistent and all models of Φ are infinite. Assume further that, for some infinite cardinal κ, any two models of Φ of cardinality κ are isomorphic. Show that Φ is then complete.

4. Let

$$\mathcal{R} = (\mathbb{R}, +, \cdot, -, 0, 1, <, =)$$

be the usual structure of the real number system, *i.e.* the ordered ring of real numbers. (a) Show that any set of real numbers, $X \subseteq \mathbb{R}$, which is first over definable over $\mathcal{R}$, consists of finitely many intervals. (b) Show that any subset of Euclidean n-space, $X \subseteq \mathbb{R}^n$, which is first-order definable over $\mathcal{R}$, consists of finitely many connected components.