Topology/Geometry Ph.D. Examination Spring 1998

Solving two problems from each section guarantees you a passing grade. Solving more problems may increase your score. You may receive partial credit for a problem **only** if you made substantial progress towards a solution. Please justify all statements you make. You may use all theorems from Math 527/528, all homework problems for these courses, and all *standard theorems* (which can be found in the *recommended books*) without proving them. You are cautioned that dropping or adding one assumption may dramatically change the difficulty of a problem. GOOD LUCK!.

Point-Set Topology

1. Let $f : X \to Y$ be a continuous closed map with the property that the inverse image of each point of Y is a compact subset of X. Show that $f^{-1}(K)$ is compact whenever K is compact.

2. Assume that Y is a Hausdorff space. Prove that if a map $f : X \to Y$ is continuous then the graph $\Gamma_f = \{(x, y) | y = f(x), \forall x \in X\}$ is closed in $X \times Y$.

3. Assume that f is a continuous one-to-one map from the figure eight to itself. Show that f must have a fixed point. What about if the assumption of one-to-one is dropped?

4. Construct an example of topological space on which every point has an open neighborhood homeomorphic to $\mathbb{R}^1$, and which is not Hausdorff.

Algebraic Topology

5. Let X be a two-torus with two holes. Compute the homology $H_*(X)$ of X.

6. Let $Y = \mathbb{C} \times \mathbb{C} - \Delta$, where Δ is the diagonal. I.e., $Y = \{(z,w) \in \mathbb{C} \times \mathbb{C} | z \neq w\}$. Let $X = Y/S_2$, where the symmetric group S_2 acts on Y by permutation. Find the fundamental group of X.

7. Let $f : S^3 \to T^3$ be a continuous map. Find $f_* H_i(S^3)$, $i = 0, \cdots 3$.

8. Prove that S^2 is not a topological group.

Differential Geometry

9. Let M be a differential manifold. Prove that T^*M is always orientable.

10. Let α be a closed two-form on S^4. Prove that $\alpha \wedge \alpha$ must vanish at some point.

11. Let $X = \mathbb{R}^3 - \{(0,0,z)\}$. Prove that the following equation:

$$(2 - (x^2 + y^2)^{\frac{1}{2}})^2 + z^2 = 1, \quad (x,y,z) \in X,$$

defines an embedding submanifold of X.

12. Let M be an n-dimensional manifold, X a complete vector field on M with flow ϕ_t. Let $\omega \in \Omega^n(M)$ be any n-form. Prove that $\phi_t^* \omega = \omega$ iff $i_X \omega$ is closed.