Topology/Geometry Ph.D. Examination Spring 1997

Solving two problems from each section guarantees you a passing grade. Solving more problems may increase your score. You may receive partial credit for a problem **only** if you made substantial progress towards a solution. Please justify all statements you make. You may use all theorems from Math 527/528, all homework problems for these courses, and all *standard theorems* (which can be found in the *recommended books*) without proving them. You are cautioned that dropping or adding one assumption may dramatically change the difficulty of a problem. GOOD LUCK!.

Point-Set Topology

1. Let $X_1 \supset X_2 \supset X_3 \dots$ be an infinite sequence of connected and closed subsets of $\mathbb{R}^2$. Is it true that $\bigcap_{i=1}^{\infty} X_i$ is connected? What if we assume that X_1 is compact?

2. Let X be the union of circles

$$\{(x,y) \in R^2 : (x-1/n)^2 + y^2 = (1/n)^2, n = 1,2,3,\dots\}$$

equipped with the subspace topology and $Y = \mathbb{R}/\rho$, where the equivalence relation ρ is given by $x \,\rho\, y$ if and only if $x = y$ or both x and y are integers. Let Y be equipped with the quotient topology. Are X and Y homeomorphic? Justify your answer.

3. Let X be a compact space and Y be a connected Hausdorff space. Let $f : X \to Y$ be a surjective continuous map such that for every point $y \in Y$, its pre-image $f^{-1}(y)$ is connected. Does this imply that X is connected?

4. Consider a topology τ on $\mathbb{R}$ whose base is given by the semi-open intervals $(a,b]$, $a, b, \in \mathbb{R}$. Determine which types of intervals $(a,b), (a,b], [a,b)$, and $[a,b]$, where $a \in \mathbb{R} \cup \{-\infty\}$ and $b \in \mathbb{R} \cup \{\infty\}$ are open and which are closed in $(\mathbb{R}, \tau)$.

Algebraic Topology

5. Consider a space X obtained by removing from $\mathbb{R}^3$ the union of the z-axis and the circle $\{z = 0, x^2 + y^2 = 1\}$. Compute the fundamental group of X.

6. Consider a manifold M whose universal cover is an even-dimensional sphere. Prove that the fundamental group of M is either trivial or isomorphic to $\mathbb{Z}_2$.

7. Construct a compact topological space X such that $H_0(X) = \mathbb{Z}$, $H_3(X) = \mathbb{Z}_5$, $H_5(X) = \mathbb{Z}$ and all other homology groups $H_k(X) = 0$.

8. Let M be a connected 5-fold cover of the orientable surface obtained from an 8-gon by the following identifications of its boundary:

Compute all the homology groups of M.

Differential Topology

9. Let $f : M \to \mathbb{R}$ be a smooth function defined on a smooth compact manifold M ($dim\ M > 1$) with exactly 2 critical points, i.e., points where the differential df identically vanishes. Prove that the fundamental group $\pi_1(M) = 0$.

10. Consider two functions $f, g : \mathbb{R}^n \to \mathbb{R}$ such that the form $df \wedge dg$ never vanishes. Prove that $V = \{x \in \mathbb{R}^n : f(x) = g(x) = 0\}$ is a smooth manifold.

11. Let ω^t and ρ^t be the one-parameter groups of diffeomorphisms of $\mathbb{R}^3$ defined by clockwise rotation around the $x-$axis and clockwise rotation around the $y-$axis, respectfully, by angle t. Let V and W be two generating vector fields for the one-parameter groups ω^t and ρ^t. Compute the bracket vector field $[V, W]$.

12. Consider the 2-form

$$\omega = \frac{x_1 dx_2 \wedge dx_3 - x_2 dx_1 \wedge dx_3 + x_3 dx_1 \wedge dx_2}{(x_1^2 + x_2^2 + x_3^2)^{\frac{3}{2}}}.$$

Let B be the closed ball $B = \{(x, y, z) : x^2 + y^2 + z^2 \leq 1\}$ in $\mathbb{R}^3$, and $F : B \to \mathbb{R}^3$ be a smooth map. Let $i(F) = \int_{\partial B} F^* \omega$. Prove that $i(F)$ is well defined, and that if $i(F)$ is not equal to 0 then the image of F contains the origin $(0, 0, 0)$.