QUALIFYING EXAM – GEOMETRY AND TOPOLOGY

May 12, 1993

Do six of the following eight problems in order to get full credit. Give complete proofs or justifications for each statement you make.

(1) Let X be a metric space. Assume that $x \in X$ is such that the space $X \setminus \{x\}$ is compact. Prove that the one element set $\{x\}$ is open in X and is a connected component of X.

(2) Compute the fundamental group of the connected sum of n tori. Recall that the connected sum of two surfaces was defined by cutting and removing a small disc from each, and then sewing the two surfaces together along the boundaries of the cuts.

(3) Let G be a topological group which is locally path connected. Let $e \in G$ be the identity. Suppose $p : \tilde{G} \to G$ is a covering map, and $\tilde{G}$ is path connected. Choose any $\tilde{e} \in \tilde{G}$ such that $p(\tilde{e}) = e$. Show that there is a unique group structure on $\tilde{G}$ such that $\tilde{e}$ is the identity, $\tilde{G}$ is a topological group, and also p is a group homomorphism.

(4) Discuss the following two statements in as much detail as you can:
  [a] Let D^n be the closed unit ball in $\mathbb{R}^n$. Every continuous map $f : D^n \to D^n$ has a fixed point.
  [b] Let S^n be the n-sphere. Every continuous map $f : S^n \to S^n$ has a fixed point.

(5) State the Poincaré duality theorem.

(6) If one removes the requirement of a differentiable manifold being Hausdorff, does a partition of unity associated to an open cover always exist? Explain your answer.

(7) Let $k \leq n$. Let $M = \{(x_1, \dots, x_n) \in \mathbb{R}^n; x_1^2 + \cdots x_k^2 > 0\}$. Compute the singular homology groups of X.

(8) Let $M = \{(x_1, \dots, x_n) \in \mathbb{R}^n; x_1^2 + x_2^2 \neq 1\}$. Compute the de Rham cohomology groups of M.