May 15, 1991

Topology And Geometry Qualifying Exam

$\mathbb{C}$ = The space of complex numbers.

$\mathbb{C}^n$ = Complex n-dimensional space (of dimension $2n$ as a smooth manifold).

$\mathbb{R}$ = The space of real numbers.

$\mathbb{R}^n$ = Euclidean space of n dimensions.

$\S^n$ = The n-dimensional sphere.

$\mathbb{C}P_n$ = Complex n-dimensional projective space (of dimension $2n$ as a smooth manifold).

Do six our of the following seven problems.

1. Let L_1 and L_2 be two disjoint lines in $\mathbb{R}^3$. Find:
   a. The fundamental group of $\mathbb{R}^3 - L_1 - L_2$.
   b. The singular homology groups of $\mathbb{R}^3 - L_1 - L_2$.
   c. Generators for the de Rham cohomology groups of $\mathbb{R}^3 - L_1 - L_2$.

2. a. Describe, with some justification, the de Rham cohomology ring of complex projective space $\mathbb{C}P_n$ of complex dimension n.
   b. Use this description to prove that any smooth map $f : S^{2n} \to \mathbb{C}P_n$ induces the zero map in dimension $2n$; that is, $f^* : H^{2n}(\mathbb{C}P_n) \to H^{2n}(S^{2n})$ is the zero map.

3. a. Give an example of a connected topological space which is not pathwise connected.
   b. Give an example of a complete metric space which is bounded but not totally bounded.

4. Let S^7 denote the seven-sphere,

$$S^7 = \left\{(z_1, z_2, z_3, z_4)\varepsilon\mathbb{C}^4 \middle| z_1, \bar{z}_1 + z_2, \bar{z}_2 + z_3, \bar{z}_3 + z_4\bar{z}_4 = 1\right\}$$

and let , denote the group whose elements are the treee cube roots of unity,

$$, = \left\{\omega\varepsilon\mathbb{C} \middle| \omega^3 = 1\right\}$$

   a. Prove that for $\omega\varepsilon$, and $(z_1, z_2, z_3, z_4)\varepsilon S^7$, we have $(\omega z_1, \omega z_2, \omega z_3, \omega z_4)\varepsilon S^7$.
   b. Let , act on S^7 by $\omega(z_1, z_2, z_3, z_4) = (\omega z_1, \omega z_2, \omega z_3, \omega z_4)$ and prove that $S^7/,$ is a manifold.

c. Compute the fundamental group of $S^7/,$.
d. Prove that $S^7/,$ is orientable.

5. Give an example of a topological space and an integer-valued singular cocycle of dimension d greater than zero which has non-zero class in the singular cohomology group $H^d(X; \mathbb{Z})$.

6. Let M be a smooth manifold satisfying the Second Axiom of Countability, and let $f : M \to \mathbb{R}$ be a smooth function such that the differential df does not vanish at any point of M.
   a. Show that M is not compact.
   b. Using partitions of unity, show that there exists a smooth vector field ξ such that $\langle df, \xi \rangle \equiv 1$ identically on M.

7. Let S^1 be the space of complex numbers of modulus one. Define $\alpha(u, v) = (-u, \bar{v})$. Let $G = \{\text{identity}, \alpha\}$ and define the Klein bottle K by setting $K = S^1 \times S^1/G$. Find the singular homology groups with integer coefficients of K.