Geometry and Topology
Qualifying Examination

August 21, 1992

Do six of the following eight problems for full credit. Give complete proofs and justifications for all assertions.

(1) Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be the map $f(x, y) = (x + y^3, y - x^2)$.
  (a) Compute the differential of f.
  (b) Is f a homeomorphism?

(2) (a) Let X be a Hausdorff topological space, and let G be a finite group of homeomorphisms of X. Prove that the quotient space X/G is Hausdorff.
  (b) (b) Give an example to show that when G is infinite, the quotient space X/G need not be Hausdorff.

(3) Let A and B be two closed subsets of a topological space X such that $X = A \cup B$. Let U be a subset of X. Prove that U is open in X if and only if $U \cap A$ is open in A and $U \cap B$ is open in B (here A and B have the subspace topology).

(4) Describe the fundamental group and the homology groups of the 3-dimensional torus $\mathbb{R}^3/\mathbb{Z}^3$.

(5) Justify the following statement: for X a smooth compact connected manifold of dimension n, X is orientable if and only if $H^n(X, \mathbb{R}) \neq 0$.

(6) Give an example of a covering map of degree 3 for which the only deck transformation is the identity map.

(7) State the implicit function theorem and give some applications of it.

(8) Is the vector field $x\frac{\partial}{\partial x} + 2y\frac{\partial}{\partial y} + 5z\frac{\partial}{\partial z}$ over $\mathbb{R}^3$ integrable?