

HWA1. Math 528.

1. Formulate the analogs of homological exact sequences (pair, Meyer-Vietoris) for co-homologies and try to describe the homomorphisms geometrically.
2. Prove that two examples of co-cycles considered in class are not co-homologous to zero.
3. A is a finite group. Describe $A \times Q$ and $A \times Z$.
4. Find $Z_{17} \times Z_{35}$, $Z_{66} \times Z_{96}$, $Z^5 \times Z^1 \mathbf{3}$.
5. Describe explicitly $Tor(A, B)$, where A, B are given as finite products of cyclic groups. Find $Tor(Z_{60} + Z^3, Z_{33} + Z_7)$.
6. Prove the correctness of the definition of tensor product via its universal property of lifting bilinear maps.
7. Prove that $H_2(X \times Y) = H_2(x) + H_2(Y) + H_1(x) \times H_1(Y)$, for path-connected X and Y .
8. Find homology groups of $RP^n \times RP^m$, $K \times RP^m$, and $L_m \times L_n \times K$, where K is the Klein bottle and L_m is a lens space, with coefficients $Z, Z_2, Z_2 \times Z_2, Z^6$. (For the definition of lens spaces, see old hwa...)
9. Give an example of two homotopy non-equivalent simplicial complexes with the same homology groups.
10. Define the degree of a map (from S^n to S^n , or any two fake manifolds) in terms of the number of preimages of a regular point, using simplicial approximation theorem.
11. Use simplicial approximation to prove *Hopf theorem*: Two maps of S^n into itself are homotopic if and only if they have the same degree. bigskip
12. Use simplicial approximation to prove a partial case of the Lefschetz theorem: a map from S^2 onto itself which is homotopic to identity has a fixed point.

HWA2. Math 528.

1. Represent complex projective spaces and 2-dim surfaces as CW-complexes and find incidence numbers. Find homology groups of the complex projective space and 2-dim surfaces by means of cellular homology.

Try the same for other examples: lens spaces, $O(n)$, Grassmanians etc.

2. Let Y_f be the result of attaching an $n > 1$ -dim cell e along $f : S^{n-1} = \partial e \rightarrow Y$. Prove that there are isomorphisms of homologies of Y and Y_f in all dimensions except n and $n - 1$, and

$$0 \rightarrow H_n(Y) \rightarrow H_n(Y_f) \rightarrow H_{n-1}(S^{n-1}) \rightarrow H_{n-1}(Y) \rightarrow H_{n-1}(Y_f)$$

is exact, moreover it is still exact with an arrow to zero from the rightmost group if $n > 2$.

3. Prove that a (continuous) map from S^n to itself which is not homotopy equivalence has a fixed point.

4. Investigate which of projective plains (real and complex) have the fixed point property.

5. Prove that a connected CW-complex is path-connected. Prove that it is connected iff its 1-skeleton is.

6. Prove that a CW-complex with a sub-complex contracted to a point is again a CW-complex.

7. Prove that a CW-sub-complex is a deformation retract of its neighborhood (a collar).

8. Prove that $H_n(M, M - pt) = \mathbb{Z}$, where M is an n -dim manifold.

9. Prove that every manifold is metrizable. Does it admit a complete metric?

10. Use implicit function theorem to prove that a solution M of $F(x) = 0$, $F : \mathbb{R}^m \rightarrow \mathbb{R}^n$ such that $\text{rank}(dF) = m$ on M is a smooth manifold. In particular, graphs of smooth functions are smooth manifolds.

11. Prove that a space covered by a manifold is a manifold.

12. Describe a smooth structure (coordinates) on Grassmanian manifolds.

HWA3. Math 528.

1. Prove that *any* structure of differentiable manifold on the real line \mathbb{R} or on the circle S^1 is equivalent to the standard one.
2. Prove that the group $\text{Diff}(M)$ of diffeomorphisms of any connected differentiable manifold M acts transitively on M .
3. Prove that any continuous real-valued function on a differentiable manifold can be arbitrarily well uniformly approximated by smooth functions.
4. Prove that any *complex manifold* is orientable. A complex manifold is a differentiable manifold which has an atlas of coordinate neighborhoods modeled on C^n and such that transition maps are given by holomorphic functions of n variables.
5. Prove that classical groups $O(n), SO(n), U(n), GL(n)$, groups of symmetrical and skew-symmetrical matrices carry natural structures of manifolds.
6. A continuous function $f : TM \rightarrow R_+$ is such that its restriction onto every tangent space is a norm. Assume that this function is smooth on a sub-bundle of non-zero vectors. Prove that the unit tangent bundle $UTM := \{v \in TM : f(v) = 1\}$ is a manifold.
7. Prove that a product of two manifolds is a manifold. Consider a product of two real-valued functions as a real-valued function from the product of manifolds to reals. Find a formula for its derivative.
8. Consider a distance function to a closed curve in R^3 . Prove that this function is smooth in a neighborhood of the curve outside of the curve, and all sufficiently small values of this function are regular.
9. Prove that a smooth manifold is orientable iff it admits an atlas with all positive Jacobians of changes of coordinates. Prove that this is equivalent to the existence of a cover by open sets with a frame field in each set, such that all orientations of the frames agree.
10. Prove that a manifold given by $F(x) = 0$, $F : R^n \rightarrow R^m$, with a regular value 0 for F , is orientable. Prove that a submanifold $M \subset R^n$ whose complement consists of two (non-empty) connected components, is orientable.
11. Prove that the Mobius strip can not be embedded into R^2 .
12. Prove that a boundary of a manifold is a manifold. Prove that it is orientable if the manifold is.
13. Prove that $S^m \times S^n$ is diffeomorphic to a submanifold in R^{m+n+1} .
14. Let M be a non-compact manifold. Prove that there exists an embedding $F : [0, \infty[\rightarrow M$, where the image of F is a closed subset.

15. All values of a map between two manifolds are regular. Prove that this map is a locally-trivial fibration.

16. A closed connected subset N of a manifold M admits a smooth retraction of M onto N . Show that N is a submanifold.

17. Is it true that a surjection F from a manifold M onto a manifold N , such that all points in N are regular values of F , is a covering map?

18. Prove that a connected component of a Lie group which contains e is a normal subgroup.

19. Prove that a quotient space L/M , where M is a Lie-subgroup of a Lie group L , is a manifold. Show that this is not necessarily a Lie group, (when M is not normal).

20. Prove that the tangent and cotangent bundle of any differentiable manifold are equivalent as vector bundles.

21. Define normal bundle of an embedded manifold.

22. Give an example of a non-trivial vector bundle, whose sum with a trivial vector bundle is again a trivial vector bundle.

23. Prove that the tangent bundles to S^3 and a torus are trivial, and to S^2 and a surface of higher genus are not.

24. Suppose M is a compact differentiable manifold and $f : M \rightarrow \mathbb{R}$ is a C^2 function which has one non-degenerate minimum, one non-degenerate maximum and no more critical points. Prove that M is homeomorphic to S^n .

25. Consider the group H of 3×3 upper-diagonal matrices with the units on the diagonal (the Heisenberg group). This group has natural coordinates (x_{12}, x_{13}, x_{23}) and it acts on itself by left translations. Let v_{12}, v_{13}, v_{23} be the left-invariant vector-fields on H with the values at the identity $(1, 0, 0)$, $(0, 1, 0)$ and $(0, 0, 1)$ correspondingly. Consider the two-dimensional distributions E and F on H generated by v_{12}, v_{13} and v_{12}, v_{23} correspondingly. Calculate both distributions in the natural coordinates and show that E is integrable and F is not.