MATH 528: TOPOLOGY/GEOMETRY

A.Katok

PROBLEM SET # 8

SIMPLICIAL COMPLEXES AND SIMPLICIAL HOMOLOGY

due on Thursday 2-9-95

1. Let K be an n-dimensional simplicial complex, K' its subcomplex consisting of all simplexes of dimension less than n. Calculate the relative homology groups $H_i(K, K')$.

2. Let K be a connected simplicial complex, L its zero-dimensional subcomplex ,ie L consists of several, say m, vertices of K. Prove that for $i \geq 2$, $H_i(K, L) = H_i(K)$ and $H_1(K, L) = H_1(K) \oplus \mathbb{Z}^{m-1}$.

3.Prove that the direct product of two simplicial polyhedra is a simplicial polyhedron.

4.Prove that the first homology group of the direct product of two simplicial polyhedra is isomorphic to the direct product of their first homology groups.

5. Let P be a compact convex polyhedron in $\mathbb{R}^3$; V, E and F be the numbers of its vertices, edges and faces correspondingly. Prove the *Euler Theorem*:$V - E + F = 2$
Hint: You may use the fact that homology groups of a simplicial polyhedron are independent of a simplicial decomposition.

6. Let K be an n-dimensional simplicial complex with the following property: the union of interiors of its $n-$ and $(n-1)-$ dimensional simplexes is connected and every $(n-1)-$ dimensional simplex belong to at mos two $n-$ dimensional simplexes Prove that $H_n(K)$ is equal to either 0 or $\mathbb{Z}$.

7. Under the conditions of the previous problem show that $H_n(K) = 0$ if one of the following conditions holds (i) the total number of $n-$ and $(n-1)-$ simplexes in K is infinite; or (ii) some $(n-1)-$ simplex belongs to the boundariy of exactly one of n simplexes. Give an example when none of these conditions hold but still $H_n(K) = 0$.

8. Calcutate the Euler characteristic of the sphere with m handles by counting the numbers of simplexes in a simplicial decomposition (cf. Problem 51). Show that for the second homology group the second alternative of problem 6 holds. Use all this to calculate the first Betti number without any direct consideration of one-cycles.

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ADDITIONAL PROBLEMS

A1. Prove that every n-dimensional simplicial polyhedron can be homeomorphically (in fact, piece-wise linearly) embedded into $\mathbb{R}^{2n+1}$.

A2. Calcutate the fundamental group of the sphere with m handles and use this calculation to calculate the first homology group.

A3. Prove that every finitely presented group is isomorphic to the fundamental group of a finite two-dimensional simplicial polyhedron. *NOTE:* Do this problem only if you did not read the section of Rothman on Seifert-Van Kampen Theorem which contains the proof of Theorem 7.45.

A4. Let P be a compact convex polyhedron in $\mathbb{R}^n$, F_m, $m = 0, 1, \ldots, n-1$ be the numbers of its m-dimensional faces. Prove the *Generalized Euler Theorem*:

$$\Sigma_{m=0}^{m=n-1} F_m = 1 + (-1)^m.$$

Hint: You may use the fact that homology groups of simplicial polyhedron are independent of a simplicial decomposition.

PROBLEM SET # 9

DEGREE, CELLURAR COMPLEXES AND CELLULAR HOMOLOGY

due on Tuesday 2-28-95

9. Every continuous map $f: S^n \to S^n$ such that $|degf| \neq 1$ has a fixed point.

10. Every continuous map $f: \mathbb{R}P(2n) \to \mathbb{R}P(2n)$ has a fixed point.

11. Give a detailed proof that for every $m \in \mathbb{Z}$ there exists a simplicial decomposition of the n–sphere S^n and a simplicial map $\phi: K \to K$ of the corresponding simplicial complex K of degree m.

Hint: Use induction and the construction of S^n as the "double cone" over S^{n-1}.

12. Prove that every finite one–dimensional CW complex allows a simplicial decomposition.

13. Let X be a CW complex which has a_k cells in dimension $2k$, $k = 0, 1, \ldots,$ Calculate homology groups of X.

14. Calculate homology groups of the Cartesian product of $S^m \times S^n$ using a cellular decomposition.

15. Let X be the set of all unit tangent vectors to the sphere S^2 with the natural topology induced from the embedding of S^2 into $\mathbb{R}^3$ (the unit tangent bundle). Prove that X allows a cellular decomposition and calculate its homology.

16. Find three linearly independent unit vector fields on S^3. Use this fact to calcutate the homology groups of the unit tangent bundle to S^3.

17. Consider the following CW complex : its 1-skeleton is the circle S^1 with the standard cellular decomposition; there are m two-dimensional cells $C_1, \ldots, C_m$ and the identification of ∂C_k with the 1-skeleton is given by the rotation by $\frac{2\pi k}{m}$. Calculate homology groups of this complex.

ADDITIONAL PROBLEMS

A5. Let $p_n : S^n \to \mathbb{R}P(n)$ be the standard projection. Prove that for $n \geq 2$ no continuous map $f : \mathbb{R}P(n) \to \mathbb{R}P(1)$ can be lifted to a map $F :, S^n \to S^1$ such that $p_1 \circ F = f \circ p_n$

A6. Construct an example of a finite two-dimensional CW complex which does not allow a simplicial decomposition.

A7. Prove that the fundamental group of a cell polyhedron is the same as for its 2-skeleton. *Note:* Carefully justify any approximation you are going to use.

A8. Use simplicial approximation to prove *Hopf theorem:* Two maps of S^n into itself are homotopic if and only if they have the same degree.

PROBLEM SET # 10

HIGHER HOMOTOPY GROUPS AND LOCALLY TRIVIAL FIBRATIONS

due on Thursday 03-16-95

18.Write down explicit formulas for the homotopies establishing commutativity of the homotopy groups $\pi_n(X, x_0)$ for $n \geq 2$ and the relative homotopy groups $\pi_n(X, A, x_0)$ for $n \geq 3$.

19.Prove that homotopically equivavent spaces have isomorphic homotopy groups. *Note:* Pay attention to the fact that the homotopy equivalences may not fix the base points.

20.Prove that all higher homotopy groups of the bouquet of $n \geq 1$ circles are trivial.

21.Calculate the higher homotopy groups of the Klein bottle.

22. Consider a fibration with the total space X, base B and fiber F. Suppose one of the three spaces is contractibe and you know homotopy groups of one of the other two. Show how to find the homotopy groups of the remainig space.

23. Prove that $\pi_3(S^2)$ is an infinite group. *Hint: Use Hopf fibration.*

24. Prove that $\pi_k(\mathbb{C}P(n)) = O$ for $3 \leq k \leq 2n$ and that $\pi_2(\mathbb{C}P(n))$ and $\pi_{2n+1}(\mathbb{C}P(n))$ are infinite groups.

25. Consider the unit tangent bundle of the sphere S^2 as locally trivial fibration with the base S^2 and the fiber S^1. Prove that the map $\Delta : \pi_2(S^2) \to \pi_1(S^1)$ in the exact sequence on this fibration has non-trivial image.

26. A *Serre fibration* is a map $p : X \to B$ for which the lifting homotopy principle holds. Give an example of a Serre fibration where both X and B are compact connected metrizable spaces and which is not a locally trivial fibration.

ADDITIONAL PROBLEMS

A9. Prove that any locally trivial fibration whose base is a disc D^n is equivalent to the direct product.

A10. Calculate homotopy groups of the sphere with n handles.

A11. Give an example of a path-connected compact metric space all of whose homotopy groups are trivial and which is not contractible.

A12.(P.Foth) Prove that the tangent bundle to the direct product of spheres where at least one sphere has an odd dimension is eqiuvalent to the direct product. In other words, if the dimension of our product space X is equal to n there are n linearly independent continuous vector fields on X.

PROBLEM SET # 11

DIFFERENTIABLE MANIFOLDS

due on Tuesday 4-4-95

27. Describe the structure of differentiable manifold on the complex projective space $\mathbb{C}P(n)$ by explicitely defining coordinate charts and calculating transition functions.

28. Give a detailed description of the structure of differentiable manifold on the sphere with n handles represented as a regular $4n$–gon with properly identified pairs of sides. Use the outline given in class on March 16.

29. Prove that *any* structure of differentiable manifold on the real line $\mathbb{R}$ or on the circle S^1 is equivalent to the standard one.

30. Prove that the group Diff(M) of diffeomomorphisms of any connected differentiable manifold M acts transitively on M. *Hint:* First prove the required property locally.

31. Prove that the group $SO(3)$ of orthogonal 3×3 matrices with determinant one is an imbedded submanifold of the nine-dimensional Euclidean space of all 3×3 matrices.

32. Prove that $SO(3)$ with the differentiable structure described in the previous problem is diffeomorphic to the real projective space $\mathbb{R}P(3)$ with the standard differentiable structure. *Hint:* Represent an ortogonal transformation as a rotation around an axis.

33. Let M be a differentible manifold and $f : M \to M$ a diffeomorphism. Consider the direct product $M \times [0,1]$ with the identification of pairs of points $(0, f(x))$ and $(1, x)$ for all $x \in M$. Show that the resulting object which we denote M_f possesses a natural structure of differentiable manifold (suspension construction). Prove that M_f is a locally trivial fibration with base S^1 and the fiber M.

34. Apply suspension construction to the following three cases :
(i) $M = \mathbb{R}, f(x) = -x$, (ii) $M = S^1, f(z) = -z$, (iii) $M = S^1, F(z) = \overline{z}$.
Identify resulting manifolds. In which of theses cases the fibration described in the previuos problem turns out to be trivial?

35. Prove that any continuous real-valued function on a differentiable manifold can be arbitrary well uniformly approximated by C^∞ functions.

PROBLEM SET # 12

VECTOR FIELDS, DISTRIBUTIONS, DIFFERENTIAL FORMS

due on Tuesday 4-25-95

36. Prove that on any non-compact connected differentaible manifold there exists an incomplete smooth vector field. *Hint:* Use partition of unity.

37. Construct three linear lenearly independent non-vanishing vector fields on S^3 and calculate their Lie brackets.

38.Consider the group H of 3×3 upper-diagonal matrices with the units on the diagonal (the Heisenberg group). This group has natural coordinates $(x_{12}, x_{13}, x_{,23})$ and it acts on itself by left translations. Let $v_{12}, v_{13}, v_{,23}$ be the left-invariant vector-fields on H with the values at the identity $(1,0,0)$, $(0,1,0)$ and $(0,0,1)$ correspondingly. Consder the two-dimensional distributions E and F on H generated by v_{12}, v_{13} and $v_{12}, v_{,23}$ correspondingly. Calculate both distributions in the natural coordinates and show that E is integrable and F is not.

39. Suppose M is a compact differentiable manifold and $f : M \to \mathbb{R}$ is a C^2 function which has one non-degenerate minimum, one non-degenerate maximum and no more critical points. Prove that M is homeomorphic to S^n.

40. Prove that the tangent and cotangent bundle of any differentiable manifold are eqivalent as vector bundles. Find a proper generalization of this statement to tensor bundles.

41. Prove that for the tangent bundle $T(M)$ one can always find another vector bundle E over M such that the Whitney sum of $T(M)$ and E is a trivial bundle *Hint:* Use an embedding theorem.

42. Prove that there is no non-vanishing skew-symmetric $2n$ differential form (a volume element) on the the real projective space $\mathbb{R}P(2n)$, $n \geq 1$.

43. Construct volume elements (non-vanishing skew-symmetric differential forms of maximal dimension) on odd-dimensional real projective spaces and all complex projective spaces.

44. Use the definition of the Lie derivative for a tensor field Ω to calculate the Lie derivative of $\rho\Omega$ where ρ is a scalar differentiable function.

PROBLEM SET # 13

You do not have to return written solutions

RIEMANNIAN METRICS, EXTERIOR DERIVATIVES, ORIENTABILITY, DE RHAM COHOMOLOGY

45.Prove that the topology defined by the distance function generated by a Riemannian metric on a differentiable manifold coincides with the topology of the manifold.

46.Prove that the metric defined by any Riemannian metric on a compact differentiable manifold is complete. Prove that on any non-compact connected differentaible manifold M there exists a Riemanmnian metric which deternines an incomplete metric on M.

47.Consider the standard embedding of the n-dimensional sphere S^n into R^{n+1} with the Riemannian metric induced by the embedding. Show that for any two points $x, y \in S^n$ which are not diametrically opposite there is a unique shortest curve in S^n connecting x and y, namely the shorter arc of the big circle. *Hint:* Use "geographical coordinates" on S^2 and induction in dimension.

48. Let ω be a non-vanishing differential 1-form. Prove that if $d\omega = \omega \wedge \alpha$ for some 1-form α then the codimension-one distribution Kerω is intergrable. *Hint:* Use Frobenius Theorem.

49. A volume element Ω on an n-dimensional manifold M determines a duality between differential $n-1$ forms and vector fields on M via interior differentiation. Prove that closed forms correspond exactly to vector fields preserving Ω, ie divergence-free vector fields.

50. Prove that any *complex manifold* is orientable. A complex manifold is a differentiable manifold which has an atlas of coordinate neighborhoods modeled on C^n and such that transition maps are given by holomorphic functions of n variables.

51. For what values of m and n the space $\mathbb{R}P(m) \times \mathbb{R}P(n)$ is orientable?

52. Describe a basis in the first cohomology group of the sphere with n handles, ie constuct $2n$ closed 1-forms $\omega_1, \ldots, \omega_{2n}$ whose cohomology classses form a basis in the cogomology group. Descbibe the multiplicative structure in the cohomology ring.*Hint:* You may (but do not have to) use Problem 49.

ADDITIONAL PROBLEMS

A13. Suppose M and N are differentiable manifolds and $f : M \to N$ is a bijection. Prove that f is a diffeomorphism if and only if both f and its inverse carry differentiable functions into differentiable functions.

A14. Prove existence of a Hausdorff connected non-separable differentiable manifold.