MATH 527: TOPOLOGY/GEOMETRY
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Homework 1 (due Friday August 8)

1. Prove that the family $\mathcal{B} = \{p_{\alpha_1}^{-1}(U_{\alpha_1}) \cap p_{\alpha_2}^{-1}(U_{\alpha_2}) \cap \ldots \cap p_{\alpha_n}^{-1}(U_{\alpha_n})\}$ forms a basis for the product topology of $X = \prod X_\alpha$.

2. Show that the family of sets $\{[a, b] \times [c, d], a, b, c, d \in \mathbb{Q}\}$ is a basis for the topology of $\mathbb{R}^2$.

3. Prove that the metrics $d_p(x, y) = ((x_1 - y_1)^p + (x_2 - y_2)^p + \ldots + (x_n - y_n)^p)^{1/p}$, $X = (x_1, x_2, \ldots, x_n)$, $p \geq 1$, define the same topology on $\mathbb{R}^n$.

4. Show that any connected component of a locally connected space is open.

5. Prove the following theorem: *A space is locally connected if and only if the family of open connected subsets is a base for the topology.*

6. Check that $\overline{A \cup B} = \overline{A} \cup \overline{B}$

7. Show that if A is a dense subset of the topological space X then $U \subset \overline{A \cup U}$ for any open set $U \subset X$.

Homework 2 (due Monday, September 18th)

1. Prove that any Hausdorff topology on a finite set is discrete.

2. Find all topologies, up to homeomorphism, on the sets $X = \{a, b\}$ and $Y = \{a, b, c\}$.

3. Define the 'boundary' of a set A in the topological space X to be $\partial A = \overline{A} \cap \overline{A^c}$. Prove that $\partial \overline{A} \subset \partial A$ and that $\partial A \supset \partial(\overset{\circ}{A})$.

4. Let $(X_\alpha, x_\alpha, \tau_\alpha)$ be a family of 'pointed' topological spaces (i.e. a distinguished point has been chosen in each space, $x_\alpha \in X_\alpha$). Let $X = \prod X_\alpha$ and $A \subset X$ be the subset of those elements all of whose components, except finitely many, are the distinguished points. Prove that A is dense in the product space X with the product topology.

5. (Use the above notations and *results.*) Assume that all X_α are connected. Prove that X is connected as well. Can we replace 'connected' with 'path connected'?

6. Prove that a space is connected if and only if any integer valued continuous map is constant.

7. Find the connected components of the countable infinite product $\{0, 1\}^\infty$.

8. Prove that $\mathbb{R}$ and $\mathbb{R}^2$ are not homeomorphic by investigating the connectivity of the complement of a point.

9. Show the following relations in the product space $X \times Y$:

   (i) $\overline{A \times B} = \overline{A} \times \overline{B}$, and

   (ii) $A \overset{\circ}{\times} B = \overset{\circ}{A} \times \overset{\circ}{B}$.

10. Let G be a *finite* group acting on a Hausdorff space X. Prove that X/G is also Hausdorff.

Homework 3 (due Wednesday, October 4th)

1. Consider on the set $\mathbb{N}$ of natural numbers the topology whose only open and not empty sets are the complements of finite sets. Find all convergent sequences and their limits in this topology.

2. Let D be a directed set and let $(x_d)_{d\in D}$ be a net in the product space $Y = \prod_{i\in I} Y_i$. Prove that $(x_d)_{d\in D}$ is convergent to l if and only if $(x_d(i))_{d\in D}$ converges to $l(i)$ for each $i \in I$.

   (This is why the product topology is also called the topology of point-wise convergence.)

3. Prove that a function $f : X \to Y$ is continuous if and only if $(f(x_d))_{d\in D}$ converges to $f(l)$ for any net x_d in X that converges to l.

4. Let $X = A \cup B$ be a topological space satisfying $\overline{A} \cap B = A \cap \overline{B} = \emptyset$. Prove that a function $f : A \to B$ is continuous if and only if both restrictions of f to A and B are continuous.

5. Let $f, g : X \to Y$ be continuous functions to a Hausdorff space Y. Prove that the set $\{x, f(x) = g(x)\}$ is closed.

6. Let X be a regular space. Prove that two sets $\overline{\{x\}}$ and $\overline{\{y\}}$ are either disjoint or equal. Define an equivalence relation on X by letting $x \equiv y$ if, by definition, the above two sets are equal. Denote by $Y = X/\equiv$ the quotient space with the quotient topology. Prove that the canonical projection $p : X \to Y$ is both open and closed and that Y is a regular Hausdorff space.

7. Prove that a product of regular spaces is again regular.

8. Show that a function $f : X \to Y$ is continuous if and only if for each $x \in X$ and each $\epsilon > 0$ there is $\delta > 0$ such that $d_Y(f(x'), f(x)) < \epsilon$ if $d_X(x', x) < \delta$.

9. Let $f : [0, \infty) \to [0, \infty)$ be an increasing function satisfying $f(x+y) \leq f(x) + f(y)$, $f(0) = 0$ and $f(x) > 0$ for $x > 0$. Prove that $f(d(x, y)$ is a metric for any metric d on a space X. (Example $\frac{x}{1+x}$.)

Homework 4 (due Wednesday, October 18th)

1. A *topological group* $(G, \cdot, \tau)$ is a set endowed with a group structure '$\cdot$' and a topology τ such that the function

$$\nu : G \times G \ni (x, y) \longrightarrow x^{-1} \cdot y = x^{-1}y \in G$$

   is continuous.

   (i) Prove that the functions $\mu(x, y) = xy$, $j(x) = x^{-1}$ are continuous ($\mu : G \times G \to G$, $j : G \to G$).

   Let $A, B \subset G$ be two arbitrary subsets of G. We will denote by $AB = \{xy, x \in A, y \in B\}$.

   (ii) Show that for every neighborhood V of e, the identity of G, there exists a neighborhood W of e such that $WW \subset V$. Moreover we can choose $W = j(W)$.

2. Let G be a topological group as above. Define $\mathcal{U}$ to be the family of subsets of $G \times G$ with the property that they contain a set of the form $S_U = \{(x, y),\ x^{-1}y \in U\}$ where U is an open neighborhood V of e.

   Prove that $\mathcal{U}$ is a uniformity on G that defines the same topology in G as the original one.

3. Prove that $\mathbb{R}$, $\mathbb{Z}$ and $GL_n(\mathbb{R})$, the group of $n \times n$ invertible matrices with real coefficients, are topological groups.

4. Let $K \subset G$ be a compact subset of a topological group G and U an open set containing K. Show that we can find an open neighborhood of e in G such that $KV \subset U$.

5. A subset A of a topological space X is called a G_δ if it is the intersection of a countable family of open sets.

   (i) Prove that if $f : X \to \mathbb{R}$ is a continuous function then $f^{-1}(0)$ is a G_δ.

   (ii) If A is a closed G_δ in a normal topological space X then there exists a continuous function $f : X \to \mathbb{R}$ such that $A = f^{-1}(0)$.

6. Prove that a subset K of a metric space X is compact if and only if every infinite subset $A \in K$ has an accumulation point in K.

7. Let $f : K \to \mathbb{R}$ be a continuous function on a compact space such that $f(x) > 0$ for any $x \in K$. Prove that there exists $\epsilon > 0$ such that $f(x) \geq \epsilon$ for any $x \in K$.

8. Let $A, B \subset X$ be two compact subsets of a metric space. Prove that there exist $a \in A$ and $b \in B$ such that $d(A, B) = d(a, b)$.

Homework 5 (due Friday)

1. Prove that a space which is connected and locally path connected is path connected.

2. (1) If G is a topological group then $\pi_0(G)$ is a group.

   (2) If G is locally path connected then every path connected component is open and closed.

3. Denote by I the unit interval $I = [0, 1]$ with the usual topology, and let X be an arbitrary space. Define a *function* $\phi : Map(I, Map(I, X)) \to Map(I \times I, X)$ as follows:

$$\phi(f)(s, t) = f(s)(t)$$

   Prove that ϕ is a homeomorphism if all function spaces are endowed with the *compact-open topology*.

Homework 6 (due Wednesday, Nov. 15)

1. Prove that $\pi_1(X \times Y, (x_0, y_0)) \simeq \pi_1(X, x_0) \times \pi_1(Y, y_0)$.

2. Prove that if $p : E \to B$ a vector bundle then, if we identify B with the zero section of E, then B is a deformation retract of E.

3. Let G be a finite group acting on the path-connected space X. Prove that if $gx \neq x$ for any $x \in X$ and $g \neq e$, then $p : X \to X/G$ is a covering.

4. Show that if G is a topological group then $\pi_1(G, e)$ is abelian.

5. Check that $[f_n][f_m] = [f_{n+m}]$ in $\pi_1(S^1, 1)$, where $f_k(t)$ is the loop $e^{2\pi \imath k t}$.

Homework 7 (due Wednesday) All spaces are assumed to be Hausdorff and locally path-connected.

1. Prove that the fundamental group of the space X is isomorphic to the group Γ_Y of deck transformations of Y, $\pi_1(X, x_0) \simeq \Gamma_Y$, if $p : Y \to X$ is a simply connected covering. More precisely, prove that the group Γ_Y is isomorphic to a subgroup of permutations of the fiber $p^{-1}(x_0)$, and that for every deck transformation $\gamma : Y \to Y$ there exists a loop f based at x_0 such that $\gamma = \phi_f$ on $p^{-1}(x_0)$.

   (Recall that f is a deck transformation if, by definition, $p = p \circ f$. Also, ϕ_f is as defined in class: $\phi_f(x) = \tilde{f}(1)$ if $\tilde{f}$ is the lift of the loop f such that $\tilde{f}(0) = x$.)

2. Prove that every locally simply connected space X has a universal covering space Y, such that $p^{-1}(x) \simeq \pi_1(X, x_0)$ where $p : Y \to X$ is the covering map.

   (Hint. Fix x_0 in X and define Y to be set of pairs $(x, [f])$ consisting of a point in X and $[f]$ is the fixed-end points homotopy class of a path f from x_0 to x. Endow Y with the quotient topology of $Map_0([0,1], X)$. Denote by $p(x, [f]) = x$ and prove that p is a covering map.)

3. Let $p : Y \to X$ be a covering space, $p(y_0) = x_0$ the base points. Prove that $p_* : \pi_1(Y, y_0) \to \pi_1(X, x_0)$ is one-to-one (i.e. injective).

4. Let $p : Y \to X$, $q : Z \to X$ be two covering spaces of the same space X, $p(y_0) = p(z_0) = x_0$ the base points. Prove that there exists a continuous map $f : Y \to Z$ satisfying $q = p \circ f$ and $f(y_0) = z_0$ if and only if $p_*(\pi_1(Y, y_0)) \subset p_*(\pi_1(Z, z_0))$.

5. (optional) The coverings of X are classified by the conjugacy classes of subgroups of $\pi_1(X)$.

6. Let X be a bouquet of n circles (that is, X is defined as the union of n circles, disjoint except for the base points which are identified). Prove that the fundamental group of X is the free nonabelian group on n generators by constructing the universal covering space and the group of deck transformations as in class.

Homework 8

1. A subspace A of X is called a *retract* of X if there exists a continuous map $r : X \to A$ which is the identity on A, $r(a) = a$, for all $a \in A$. Prove that if A is a retract of X then
$$H_n(X) \simeq H_n(A) \oplus H_n(X, A)$$

2. Check the functorial properties of relative homology groups $H_n(X, a)$.

   (That is, check that any map of pairs $f : (X, A) \to (Y, B)$ induces a morphism $H_n(f) = f_* : H_n(X, A) \to H_n(Y, B)$ such that $H_n(f \circ g) = H_n(f) \circ H_n(g)$ and $H_n(id) = id$.)

3. If every path component of X intersects A then $H_0(X, A) = 0$.

4. Let $f_n : S^1 \to S^1$ be the continuous map $f_n(z) = z^n$. Prove that $H_1(f_n) : H_1(S^1) \to H_1(S^1)$ is the multiplication by n.

5. Consider two chain complexes $C = (C_n, \partial)_{n\geq 0}$ and $C' = (C'_n, \partial')_{n\geq 0}$ and two morphisms f, g between them. That is we assume that we have two sequences of morphisms $f = (f_n)_{n\geq 0}$, $g = (g_n)_{n\geq 0}$, $f_n, g_n : C_n \to C'_n$ satisfying
$$\partial' \circ f_n = f_{n-1} \circ \partial \ , \ \ \partial' \circ g_n = f_{n-1} \circ \partial.$$
   The morphisms f and g are called *chain homotopic* if there exists a sequence of morphisms $h_n : C_n \to C'_{n+1}$ (note the shift in degree!) such that
$$f_0 - g_0 = \partial' \circ h_0$$
$$f_n - g_n = \partial' \circ h_n + h_{n-1} \circ \partial \text{ for } n > 0$$
   Prove that two chain homotopic morphisms induce the same morphism between the homology groups of the complexes
$$f_* = g_* : H_n(C) \longrightarrow H_n(C').$$

6. Denote by X_α, $\alpha \in I$ the path components of a space X. Construct natural inclusions morphisms $H_n(X_\alpha) \to H_n(X)$. Prove that they are one-to-one and they provide an isomorphisms
$$\oplus_{\alpha \in I} H_n(X_\alpha) \simeq H_n(X)$$

and

$$\oplus_{\alpha \in I} H_n(X_\alpha, A \cap X_\alpha) \simeq H_n(X, A)$$

for any subspace $A \subset X$.

7. Let X and X' be two disjoint spaces. Choose arbitrary points $x_0 \in X$ and $x_0' \in X$. Denote by Y the space constructed from the disjoint union of X and X' with the two point identified. Construct natural morphisms $\phi : H_n(X) \to H_n(Y)$ and $\psi : H_n(X') \to H_n(Y)$. Prove that they are one-to-one for any n and that for $n > 0$ they provide us with an isomorphism $H_n(Y) \simeq H_n(X) \oplus H_n(X')$.

8. Prove that if $p : Y \to X$ is a covering space map then it induces an isomorphism of the higher homotopy groups $p_* : \pi_n(Y, y_0) \to \pi_n(X, x_0)$ for $n > 1$.