MATH 527: TOPOLOGY/GEOMETRY

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PROBLEM SET # 1

TOPOLOGICAL SPACES

Due on Monday 9-12-94

1. Find all different topologies (up to a homeomorphism) on the sets consisting of 2 and 3 elements.

2. Prove that the set of squares of rational numbers is dense in the set of all non-negative real numbers.

3. Prove that for any set A in a topological space $\partial\overline{A} \subset \partial A$ and $\partial(\operatorname{Int} A) \subset \partial A$. Give an example when all these three sets are different.

4. We say that a topological space $(X, \mathcal{T})$ satisfies (T_1) separation axiom (or simply is a (T_1)-space) if for any two different points x and y there exists an open set U which contains x and does not contain y. Prove that $(X, \mathcal{T})$ is a (T_1)-space if and only if any set consisting of one point is closed.

5. Find among examples given in class a topological space which is a (T_1)-space, but not a (T_2) (Hausdorff) space.

6. Prove that the product of countably many separable topological spaces with the product topology is separable.

7. Prove that a topological space $(X, \mathcal{T})$ is connected if and only if any continuous function from X to the set of integers (with discrete topology) is constant.

8. Prove that $\mathbb{R}$ (the real line) and $\mathbb{R}^2$ (the plane with the standard topology) are not homeomorphic. *Hint:* Use the notion of a connected set.

ADDITIONAL PROBLEMS

A1. A topological space $(X, \mathcal{T})$ is called *regular* (or (T_3)- space) if for any closed set $F \subset X$ and any point $x \in X \setminus F$ there exist disjoint open sets U and V such that $F \subset U$ and $x \in V$. Give an example of a Hausdorff topological space which is not regular.

A2. Prove that a topological space $(X, \mathcal{T})$ is connected if and only if any continuous function $f : X \to K$, where K is the Cantor set, is constant.

A3. A point x in a topological space is called *isolated* if the one-point set $\{x\}$ is open. Prove that any compact separable Hausdorff space without isolated points contains a closed subset homeomorphic to the Cantor set.

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PROBLEM SET # 2

EXAMPLES OF TOPOLOGICAL AND METRIC SPACES

NN9–12 are due on Monday 9-19-94; NN13–16 and A4–A7 are due on Monday 9-26-94

9. Find all different topologies (up to a homeomorphism) on a set consisting of 4 elements which make it a connected topological space.

10. Let X, Y be two topological spaces $f : X \to Y$ be a continuous map,

$$\operatorname{graph} f = \{(x, f(x)) \in X \times Y;\ x \in X\}.$$

Prove that $\operatorname{graph} f$ with the topology induced from $X \times Y$ is homeomorphic to X.

11. Prove that the set $[0,1] \times [0,1] \setminus K \times K$, where K is the standard Cantor set, is path-connected.

12. Let $\{t_n\},\ n = 1, 2, \ldots$ be a sequence of positive numbers such that the series $\sum_{n=1}^{\infty} t_n$ converges. Let for $\omega, \omega' \in \Omega_2$

$$d(\omega, \omega') = \sum_{n=1}^{\infty} t_n |\omega_n - \omega'_n|.$$

Prove that this formula defines a metric on the space Ω_2 which generates the product topology.

13. Consider the product $\Omega_m,\ m \geq 2$ of countably many m-point sets with discrete topology. Prove that Ω_m provided with the product topology is a Cantor space i.e. is homeomophic to the standard Cantor set K.

14. Let the group $\mathbb{R}$ act on $\mathbb{R}^2$ by

$$t(x_1, x_2) = (x_1, x_2 + x_1 t).$$

Prove that the factor-space with the factor topology is not Hausdorff but it is a union of two disjoint subsets each of which is a Hausdorff topological space. Prove also that the factor space is a T_1-space (see problem 4).

15. For a given prime number p define the *p-adic norm* $\| \ \|_p$ on the field $\mathbb{Q}$ of rational numbers by

$$\|r\|_p = p^{-m}, \text{ if } r = p^m \frac{k}{l}, \text{ where } mk, l \in \mathbb{Z} \text{ and } k \text{ and } l \text{ are relatively prime with } p.$$

Prove that the distance function $d(r, r') = \|r - r'\|_p$ defines a metric on $\mathbb{Q}$. The completion of $\mathbb{Q}$ in that metric is called *p-adic numbers* and the completion of $\mathbb{Z}$ is called *p-adic integers*. Prove that the space of p-adic integers is a Cantor space and the space of p-adic numbers is homeomorphic to the disjoint union of countably many Cantor spaces. *Hints:* Use the fact that integers lie in the unit ball around zero. Use Problem 13.

16. Define the *profinite topology* on the group $\mathbb{Z}$ of integers as the weakest topology in which any arithmetic progression is an open set. Let $\mathbb{T}^\infty$ be the product of countably many copies of the circle with the product topology. Define the map $\varphi : \mathbb{Z} \to \mathbb{T}^\infty$ by

$$\varphi(n) = (\exp(2\pi in/2), \exp(2\pi in/3), \exp(2\pi in/4), \exp(2\pi in/5), \dots)$$

Show that the map φ is injective and that the topology induced on $\varphi(\mathbb{Z})$ coincides with profinite topology.

ADDITIONAL PROBLEMS

A4. Show that the closure of $\varphi(\mathbb{Z})$ as in problem 16 is homeomoprhic to the Cantor set. Introduce a translation-invariant metric on $\mathbb{Z}$ which generates the profinite topology and such that Cauchy sequences in that metric are exactly the sequences whose images under φ converge in $\mathbb{T}^\infty$.

A5. Consider the weakest topology in the set $\mathbb{R}$ or real numbers such that for any $t \in \mathbb{R}$ the function $x \to \exp(itx)$ is continuous. Prove that this topology is not metrizable.

A6. Prove that any compact metrizable toplogical space is homeomorphic to a closed (and hence compact) subset of the *Hilbert cube*, i.e. the product of countably many unit intervals with the product topology (universality of the Hilbert cube).

A7. A metric space X is called *locally path connected* if for any $\epsilon > 0$ there exists $\delta > 0$ such that any two points at a distance less than δ can be connected by a path contained in a ball of radius ϵ

Prove that for any compact path connected, and locally path connected subset X of the plane $\mathbb{R}^2$ there exists a continuous map $f : [0, 1] \to \mathbb{R}^2$ whose image coincides with X (Generalized Peano curve).

PROBLEM SET # 3

COMPACT AND COMPLETE SPACES; TOPOLOGICAL GROUPS

due on Wednesday 10-12-94

17. Prove that any separable metric space has a countable base.

18. Prove that for a metric space compactness is equivalent to sequential compactness: Every sequence contains a converging subsequence. *Hint:* Use previous problem.

19. Let X be a compact Hausdorff space with a countable base (and hence metrizable). Prove that the topology in the space $\mathcal{F}(X)$ of all closed subsets of X induced by the Hausdorff metric does not depend on the metric in X defining the given topology.

20. A metric space X is called *precompact* if for any $\epsilon > 0$ it can be covered by finitely many ϵ-balls. Prove that the completion of a metric space X is compact if and only if X is pre-compact.

21. Let the *weak topology* in the Hilbert space $l^2(\mathbb{R})$ be the weakest topology in which all maps $f : l^2(\mathbb{R}) \to \mathbb{R}$ of the form

$$f(x) = \Sigma_{n=1}^{\infty} a_n x_n, for some\ (a_1, a_2, \dots) \in l^2(\mathbb{R}).$$

are continuous. Prove that the weak topology is weaker than the standard (norm) topology i.e that there are open sets in the norm topology which are not open in the weak topology.

22. A *topological group* is a group G endowed with a topology such that the group multiplication and taking inverse are continuous operations, i.e. the maps $G \times G \to G : (g_1, g_2) \to g_1 g_2$ and $G \to G : g \to g^{-1}$ are continuous. Two topological groups are isomorphic if there exists a group isomorphism between them which is also a homeomorphism. Prove that the space Ω_2 with the coordinate-wise modulo 2 addition as the group operation and the product topology is a topological group.

23. Consider the group $SL(2, \mathbb{R})$ of all 2×2 matrices with determinant one with the topolology induced from the coordinate embedding into $\mathbb{R}^4$. Prove that it is a topological group.

24. Prove that the addition and multiplication can be extended in a unique way from the rationals and non-zero rationals correspondingly to the set of p-adic numbers (problem 15) and non-zero p-adic numbers correspondingly so that the topology of problem 15 makes those into topological groups.

ADDITIONAL PROBLEMS

A8. Prove that any closed convex bounded set in $l^2(\mathbb{R})$ (e.g. any closed ball) is compact in weak topology.

A9. Prove that any set in $l^2(\mathbb{R})$ compact in the weak topology is closed and bounded. Give an example of a closed bounded set in $l^2(\mathbb{R})$ which is not compact in the weak topology.

A10. Give an example of a compact metrizable path-connected topological space X such that no point of X has a connected neighborhood.

A11. Consider the metric on $\mathbb{Z}$ defining the profinite topology as in Problem A4. Show that addition can be extended in a unique way to the completion to make it into a topological group which is isomprhic as topological group to the product of the groups of p-adic integers for $p = 2, 3, 5, \dots$.

A12. Consider the following subgroup S of $\mathbb{T}^\infty$, the product of countably many copies of the circle: $S = \{(z_1, z_2, z_3, \dots) : z_n^2 = z_{n-1}, n = 2, 3, \dots\}$ with the topology induced from $\mathbb{T}^\infty$. Prove that as topological space S is connected but not path-connected.

PROBLEM SET # 4

MISCELLANEOUS GENERAL TOPOLOGY

due on Wednesday 10-26-94

25. Prove that on the real line $\mathbb{R}$ there are uncountably many different (non-equivalent) complete uniform structures compatible with the standard topology *Hint:*Use different metrics.

26. Prove that for any natural number n the standard n-dimensinal simplex

$$\sigma^n = \{(x_1, \dots, x_n, x_{n+1} \in \mathbb{R}^{n+1} : \Sigma_{k=1}^{n+1} x_k = 1, x_k \geq 0, k = 1 \dots, n\}$$

is homeomorphic to the closed unit ball in $\mathbb{R}^n$

27. Consider the unit sphere in $\mathbb{R}^n$ as a homogenous space of the group $SO(n)$ of orthogonal matrices with determinant one. Prove that the factor-topology coincides with the standard topology induced from $\mathbb{R}^n$.

28. Let X be a compact Hausdorff space. Prove that the space of continous maps form X to the unit interval is compact if and only if X contains finitely many elements.

29. Prove that for any natural number k the space $C^k(\mathbb{R}^2)$ of all k times continuously differentiable functions of two real variables (with the topology of uniform convergence on compact sets of the functions and all partial derivaves of order up to k and the corresponding uniform structure) is complete.

30. Prove that the figure eight (i.e.the union of two circles with one common point) is not contractible.

31. Prove that the product of a finite or countable collection of contractible spaces is contractible.

32. *A path-connected component* of a topological space X is a maximal path-connected subset of X. Prove that any space can be decomposed in a unique way into path-connected components.

ADDITIONAL PROBLEMS

due on Wednesday 11-2-94

A13.Describe uncountably many different *incomplete* uniform structures on the real line $\mathbb{R}$ compatible with the standard topology.

A14.Let $G = SL(2, \mathbb{R})$ be the topological group of all 2×2 matices with determinant one. Consider the subrgoup $H = SL(2, \mathbb{Z}) \subset SL(2, \mathbb{R})$ of all matrices with integer entries. Prove that the homogenous space G/H with the factor-topology is normal, locally compact but not compact.

A15. Construct a continuous map from the unit interval onto Hilbert cube (Infinite-dimensional Peano curve). Try not to do an explicit construction from the scratch but use existing examples instead.

A16. Prove that in the space $C([0, 1])$ of continuous functions on the unit interval the set of functions which are monotone on some interval has first category.

A17. *A one-dimensional complex* is a topological space which consists of a finite or countable union of sets (*edges*) each of which is homeomophic to the unit interval with disjoint interiors and such that any endpoint of any edge belong only to finitely many edges. A *loop* is a collection of edges $\{E_1, \ldots, E_n\}$ such that oone end point of E_1 is also an endpoint of E_n, the other endpoint of E_1 is an endpoint of E_2, the other endpoint of E_2 is an endpoint of E_3 etc. A complex is a *tree* if it does not contain any loops. Prove that one-dimensional complex is contractible if and only if it is a tree.

A18. Describe in detail the path-connected components of the tolpological space of Problem A12 (dyadic solenoid). in particular prove that every path-connected component is dense.

PROBLEM SET # 5

HOMOTOPY EQUIVALENCE, FUNDAMENTAL GROUP

due on Friday 11-11-94

33. The (open) *Mobius strip* is the factor (orbit space) of $\mathbb{R}^2$ by the action of the group generated by integer translations along the y-axis and the transformation $T : (x, y) \to (-x, y + 1)$. Prove that the Mobius strip is homotopically equivalent to the circle.

34. Prove that any convex set in $\mathbb{R}^n$ lies inside a certain affine subspace and contains an open ball in that subspace.

35. Prove that any convex set in $\mathbb{R}^n$ is contractible.

36. Prove that the fundamental group of the Carthesian product of two path-connected topological spaces is isomorphic to the direct product of their fundamental groups.

37. Find the fundamental group of the figure eight.

38. Prove that any contractible space is path-connected.

40. Prove that the open cylinder with one point removed and the torus $\mathbb{T}^2$ with one point removed are homolopically equivalent and calculate the fundamental group of those spaces.

41. The *projective plane* is the factor-space of the two-dimensional sphere where pairs of opposite point are identified. Prove that the projective plane is not contractible and is not homotopically equivalent to a sphere or a torus of any dimension. *Hint:* Use fundamental groups.

ADDITIONAL PROBLEMS

due on Monday 11-28-94

A19. Prove the following special (in fact, a leading) case of the *Tychonov fixed-point theorem*: Every continuous map of the Hilbert cube into itself has a fixed point. You may use Brouwer fixed-point theorem.

A20. The *Klein bottle* is the factor (orbit space) of $\mathbb{R}^2$ by the action of the group generated by integer translations along the x-axis and the transformation $T : (x, y) \to (-x, y + 1)$. Prove that the Klein bottle is a topological manifold. Prove that it is not homotopically equivalent to the bouquet of $n \geq 1$ circles.

A21. Prove that the fundamental group of any compact topological manifold is finitely generated.

A22. Prove that the fundamental group of any one-dimensional complex (See Problem A18) is a free group with a finite or countable number of generators

A23. Prove that the unit sphere in the Hilbert space $l^2(\mathbb{R})$ is contractible.

PROBLEM SET # 6

DEFORMATION RETRACTS, FUNDAMENTAL GROUP, COVERING SPACES.

due on MONDAY 11-28-94

42. Give a detailed rigorous argument showing that the figure eight is a strong deformation retract of the "basic pretzel" (the solid double torus).

43. Consider the quadric in $\mathbb{R}^n$ given by the equation $\Sigma_{i=1}^{k} x_i^2 - \Sigma_{i=k+1}^{n} x_i^2 = 1$. Prove that it has a $k-1$-dimensional sphere as a deformation retract.

44. Calculate the fundamental group of the topological group $SL(2,\mathbb{R})$.

45. Recall that the *complex projective space* $\mathbb{CP}(n)$ is the space of all complex lines passing through the origin in the $n+1$-dimensional complex space $\mathbb{C}^{n+1}$, or, equivalently, the factor of $\mathbb{C}^{n+1}$ minus the origin with respect to the action of the multiplicative group of non-zero complex numbers by the scalar multiplication . The natural embedding $\mathbb{C}^n \subset \mathbb{C}^{n+1}$ generates an embedding $\mathbb{CP}(n-1) \subset \mathbb{CP}(n)$. Prove that $\mathbb{CP}(n-1)$ is a strong deformation retract of $\mathbb{CP}(n)$ with one point deleted.

46. Prove that $\mathbb{CP}(n)$ is simply connected. *Hint:* Use previous problem and induction in dimension.

47. Calculate the fundamental group of the real projective space $\mathbb{RP}(n)$.

48. The *Klein bottle* is the factor (orbit space) of $\mathbb{R}^2$ by the action of the group generated by integer translations along the x-axis and the transformation $T : (x, y) \to (-x, y+1)$. Prove that the Klein bottle is a topological manifold. Prove that it is not homotopically equivalent to any of the following spaces: point, sphere of any dimension, torus of any dimension.

49. Prove that any covering space of the Klein bottle is homeomorphic to one of the following spaces: $\mathbb{R}^2$, $\mathbb{T}^2$, open cylinder, Mobius strip and Klein bottle.

50. Describe the covering space of the figure eight corresponding to the commutant of the fundamental group.

PROBLEM SET # 7

SIMPLICIAL COMPLEXES AND SIMPLICIAL HOMOLOGY

due on Friday 12-9-94

51. Describe a simpicial decomposition of the sphere with n handles.

52. Describe a simpicial decomposition of the real projective space $\mathbb{R}P^n$.

53. Calculate the first and second homology groups of the Klein bottle. Hint: Use Poincaré-Hurewicz Theorem.

54. Prove that the homology groups of a bouquet of finitely many connected simplicial complexes are direct products of the corresponding homology groups.

55. Using problem 52 calculate all homology groups of the space $\mathbb{R}P^3$.

56. Find the minimal number of verteces in a simplicial complex S such that $H_1(S) = \mathbb{Z}^6$.