Mid-Term Problems and Solutions, Spring 1996

We will use several times the result that describes the homology of an adjunction space $X = Y \cup_f e^m$. Namely, recall that the relevant morphism is $f_* : H_{m-1}(S^{m-1}) \to H_{m-1}(Y)$, and that it determines the homology groups of X as follows

$$H_q(X) \simeq H_q(Y) \text{ for } q \neq m-1, m, \tag{1}$$

$$H_{m-1}(X) \simeq H_{m-1}(Y)/Im(f_*), \tag{2}$$

$$H_m(X) \simeq H_m(Y) \oplus \ker(f_*). \tag{3}$$

An often used computation is that of $f_* = H_0(f)$ between path-connected spaces: $H_0(f) = id$.

An other frequently used fact is the vanishing of a group morphism between two groups one of if one of the groups is zero.

1. (a) Determine $H_q(S^1 \times S^1)$. Let $p_1, p_2 : S^1 \times S^1 \to S^1$ be the two projections. Find the morphisms $p_{1*}, p_{2*} : H_q(S^1 \times S^1) \to H_q(S^1)$.

Proof. The torus $S^1 \times S^1$ is obtained by attaching a 2-cell to the space $S^1 \vee S^1 = S^1 \times \{1\} \cup \{1\} \times S^1$:

$$S^1 \times S^1 = (S^1 \vee S^1) \cup_f e^2.$$

The morphism $f_* : H_1(S^1) \to H_1(S^1 \vee S^1)$ was determined in class (using Hurewicz's theorem) to be zero. We obtain that the inclusion induces an isomorphism $H_1(S^1 \vee S^1) \simeq H_1(S^1 \times S^1)$ (see equation (2) above). This also determines $H_0(S^1 \times S^1) \simeq H_2(S^1 \times S^1) \simeq \mathbb{Z}$, $H_2(S^1 \times S^1) \simeq \mathbb{Z}^2$, and that all the other groups vanish.

Recall that the isomorphism $H_1(S^1 \vee S^1) \simeq \mathbb{Z} \oplus \mathbb{Z}$ is obtained from the two inclusions $j_1, j_2 : S^1 \to S^1 \vee S^1$. The explicit formulas for these inclusions are $j_1(z) = (z, 1)$ and $j_2(z) = (1, z)$. The isomorphism that results is

$$j_{1*} \oplus j_{2*} : H_1(S^1) \oplus H_1(S^1) \to H_1(S^1 \vee S^1).$$

Thus, in order to compute p_{i*} on H_1 it is enough to find $p_{i*} \circ j_{k*}$. Since for $i = k$ the composition $p_i \circ j_k$ is the identity and for $i \neq k$ it is a constant we obtain that $p_{1*}(a, 0) = p_{1*}\big(j_{1*}(a)\big) = a$ and $p_{1*}(0, b) = p_{1*}\big(j_{2*}(b)\big) = 0$. Summing up $p_{1*}(a, b) = a$ is the projection onto the first component. Similarly $p_{2*}(a, b) = b$ is the projection onto the second component.

On H_0 both p_{i*} are the identity (after canonical identifications, see above) and on the other groups are zero (because $H_q(S^1) = 0$ for $q > 1$).

□

(b) Let m, n be two integers and define $\phi : S^1 \to S^1 \times S^1$ by the formula $\phi(z) = (z^n, z^m)$. Determine ϕ_*.

Proof. We observe as above that ϕ_* is the canonical isomorphism on H_0, and vanishes on H_q for $q > 2$. The nontrivial morphism is thus the one defined on H_1. We know that

$$\phi_*(k) = (ka, kb) \in \mathbb{Z} \oplus \mathbb{Z} \simeq H_1(S^1 \times S^1)$$

for some integers a, b to be determined. The formulas for p_{i*} above tell us that $p_{1*}(\phi_*(k)) = p_{1*}(ka, kb) = ka$. On the other hand $p_1 \circ \phi(z) = z^n$ is a map of degree n (proved in class) so $a = n$. Similarly $b = m$.

□

(c) Prove that $\tau_* = -1$ on $H_2(S^1 \times S^1)$ if $\tau(z, w) = (w, z)$.

Proof. ¿From the first part we know that the inclusion gives an isomorphism $H_2(S^1 \times S^1) \simeq H_2(S^1 \times S^1, S^1 \vee S^1)$. Since τ induces a map of the pair $(S^1 \times S^1, S^1 \vee S^1)$ it is enough to determine the action of τ_* on $H_2(S^1 \times S^1, S^1 \vee S^1) \simeq H_2(S^1 \times S^1 / S^1 \vee S^1)$. We can choose a homeomorphism $\psi : S^1 \times S^1 / S^1 \vee S^1 \simeq S^2$ that will transform the map τ into the map $\tau'(x, y, z) = (y, x, z)$ (i.e. $\tau' = \psi \tau \psi^{-1}$). The morphisms τ and τ' will have the same degree. Since τ' comes from a linear map, its degree is the sign of the determinant, that is -1.

(An argument closer to one of the homeworks would be to say that τ' is the suspension of the map $z \to i\overline{z}$ which is homotopic to the map $z \to \overline{z} = z^{-1}$ on the unit circle.)

□

For any integer $k \in \mathbb{Z}$ we define $Z_k = D^2/\equiv$, where $D^2 = \{z \in \mathbb{C}, |z| \leq 1\}$, and $z \equiv w$ if and only if $z = w$ or $z^k = w^k$ and $|z| = 1$. Fix two integers m and n.

(a) Prove that the spaces $X = Z_m$ and $Y = Z_m \times Z_n$ are CW-complexes by providing explicit realizations. (i.e. find the k-skeleta $X_0 \subset X_1 \subset X_2 = X$ and $Y_0 \subset Y_1 \subset \ldots \subset Y_4 = Y$ and describe the cells that are attached and the attaching maps.)

Proof. We denote by $f_k(z) = z^k$. Then $X_0 = e^0$, $X_1 = e^0 \cup e^1 = S^1$ and $X^2 = S^1 \cup_{f_m} e^2$. This describes the CW-complex structure of the space X.

Denote $X' = Z_n$ and write X' similarly as $X' = f^0 \cup f^1 \cup_{f_n} f^2$ (the cells of X' are denoted by the letter f).

We have that the skeleta of Y is given by

$$Y_k = \bigcup_{i+j=k} X_i \times X_j.$$

This gives one 0-cell, two 1-cells, three 2-cells ($e^0 \times f^2$, $e^1 \times f^1$ and $e^2 \times f^0$), two 3-cells and one 4-cell.

The attaching maps of the 2-cells are $id \times f_n$, f-defined in the first problem-and $f_m \times id$. The second attaching map appears from the 2-cell of the torus $S^1 \times S^1 \subset Y$.

□

(b) Determine the chain complexes $C(X)$ and $C(Y)$ associated to the above CW-complexes.

(c) Compute the groups $H_q(X)$ and $H_q(Y)$.

We will assume that $m, n \neq 0$. The same reasoning applies to the case when one of m or n vanishes, but the results will have a different form.

Proof. We have from the definition $C(X) = (C_q(X), d_q)$ where $C_q(X) = H_q(X_q, X_{q-1})$ and $d_q = j_* \circ \partial$, defined on $C_q(X)$, where $j : X_{q-1} \to (X_{q-1}, X_{q-2})$ is the inclusion of pairs. This gives for the space X, $d_1 = 0$ and $d_2(e^2) = me^1$ (we identify the cells with the generators of the groups). We obtain $H_0(X) \simeq \mathbb{Z}$, $H_1(X) \simeq \mathbb{Z}_m$ and $H_q(X) \simeq 0$ for $q > 1$.

For the space Y we have on generators

$$d_1(e^1 \times f^0) = d_1(e^0 \times f^1) = 0,$$

$$d_2(e^2 \times f^0) = m(e^0 \times f^1)\,,\ d_2(e^1 \times f^1) = 0$$

(like for the torus), and

$$d_2(e^0 \times f^2) = n(e^0 \times f^1).$$

Similarly (with a grain of trust...)

$$d_3(e^2 \times f^1) = m(e^1 \times f^1)\,,\ d_3(e^1 \times f^2) = -n(e^1 \times f^1)$$

and, finally,

$$d_4(e^2 \times f^2) = m(e^1 \times f^2) + n(e^2 \times f^1).$$

This gives $H_0(Y) \simeq \mathbb{Z}$, $H_1(Y) \simeq \mathbb{Z}_n \times \mathbb{Z}_m$,

$$H_2(Y) \simeq \mathbb{Z}/\{am - bn, a, b \in \mathbb{Z}\} \simeq \mathbb{Z}/(m,n)\mathbb{Z},$$

$$H_3(Y) \simeq \{(a,b), am - bn = 0, a, b \in \mathbb{Z}\}/\{(cn, cm), c \in \mathbb{Z}\} \simeq \mathbb{Z}_d$$

where $d = (m,n)$ is the greatest common divisor of m and n and $H_q(Y) = 0$ for $q > 3$.

□

(a) Determine the morphisms $j_* : H_q(\mathbb{C}P^n) \to H_q(\mathbb{C}P^{n+1})$ induced by the inclusion $j : \mathbb{C}P^n \to \mathbb{C}P^{n+1}$.

Proof. We know that $\mathbb{C}P^{n+1} = \mathbb{C}P^n \cup e^{2n+2}$ and hence the attaching map induces a vanishing morphism

$$0 = f_* : H_{2n+1}(S^{2n+1}) \to H_{2n+1}(\mathbb{C}P^{2n}).$$

This proves (see equations (1), (2) and (3) above) that j_* is an isomorphism for all $q \neq 2n+2$, and that it is always one-to-one.

□

(b) Compute the homology groups $H_q(\mathbb{C}P^{n+k}, \mathbb{C}P^n)$.

Proof. The inclusion $\mathbb{C}P^n \to \mathbb{C}P^{n+k}$ induces morphisms between homology groups which by functoriality coincide with the composition

$$H_q(\mathbb{C}P^n) \to H_q(\mathbb{C}P^{n+1}) \to \ldots \to H_q(\mathbb{C}P^{n+k})$$

of the morphisms from the first part of the problem. Since all above morphisms are one-to-one (i.e. injective) it follows that their composition is also one-to-one.

Using this fact in the exact sequence of the pair $(\mathbb{C}P^{n+k}, \mathbb{C}P^n)$ we obtain that

$$H_q(\mathbb{C}P^{n+k}, \mathbb{C}P^n) \simeq H_q(\mathbb{C}P^{n+k})/H_q(\mathbb{C}P^n).$$

This finally gives

$$H_q(\mathbb{C}P^{n+k}, \mathbb{C}P^n) \simeq 0/0 = 0 \text{ for } q \text{ odd or } q > 2(n+k),$$

$$H_q(\mathbb{C}P^{n+k}, \mathbb{C}P^n) \simeq \mathbb{Z}/\mathbb{Z} = 0 \text{ for } q = 0, 2, \ldots, 2n$$

and

$$H_q(\mathbb{C}P^{n+k}, \mathbb{C}P^n) \simeq \mathbb{Z}/0 = \mathbb{Z} \text{ for } q = 2(n+1), \ldots, 2(n+k)$$

(the only non-vanishing groups).

□

(c) Determine the morphism $l_* : H_q(\mathbb{R}P^n) \to H_q(\mathbb{C}P^n)$ determined by the natural inclusion, $l : \mathbb{R}P^n \to \mathbb{C}P^n$,

$$l([x_0, x_1, \dots, x_n]) = [x_0, x_1, \dots, x_n].$$

Proof. On H_0 we know that $l_* = id$ because both spaces are path-connected. All other morphisms are 0. This is because $H_q(\mathbb{R}P^n) = 0$ for $q > 0$ even and $H_q(\mathbb{C}P^n) = 0$ for q odd. □