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MID-TERM EXAMINATION

MATH 528: TOPOLOGY/GEOMETRY

SIMPLICIAL AND CELL HOMOLOGY

Saturday 3-4-95

Do one problem from Section 1 and three problems from Section 2.

SECTION 1

1.1.Prove that if $m < n$ then any continuous map $f : S^m \to S^n$ is null-homotopic, ie homotopic to a map into a point.

1.2. Prove that there is no continuous map $f : S^n \to S^1$ for $n > 1$ which sends the opposite points into opposite points, ie $f \circ I_n = I_1 \circ f$ where I_k is the flip map on S^k.

1.3. Prove using the degree theory for the maps of the sphere that any non-constant polynomial with complex coefficients has a complex root.

SECTION 2

2.1.Let K be a finite n-dimensional simplicial complex with the following property: the union of interiors of its n– and $(n-1)$– dimensional simplexes is connected and every $(n-1)$– dimensional simplex belongs to exactly two n– dimensional simplexes. Let L be the complex obtained from K by eliminating one of n–dimensional simplexes. Suppose you know the Betti numbers of K.

Describe all possibilities for the Betti numbers of L.

2.2. Calculate homology groups of $S^2 \times \mathbb{R}P(2)$.

2.3.Consider the three–dimensional torus represented as the unit cube in $\mathbb{R}^3$ with pairs of opposite faces identified. Consider the group of order three generated by the rotation by $\frac{2\pi}{3}$ around one of the main diagonals. The factor–space possesses a natural cellular decomposition which is inherited from the standard decomposition of the torus.

Calculate the cellular homology of the factor–space.

2.4. Consider the space of oriented big circles in S^3 or, equivalently, the space of oriented two-dimensinal subspaces of $\mathbb{R}^4$ with the natural topology.

Construct a cell decomposition of this space and calculate its homology groups.

2.5. Consider the following subset S of $\mathbb{R}^3$:

$$S = \{(x, y, z) \in \mathbb{R}^3 : \ ((x^2 + y^2)^{\frac{1}{2}} - 1)^2 + z^2 = 1\}.$$

In other words, S is the surface of revolution around z axis of the circle in the xz plane with the center on the x axis which passes through the origin.

Construct a cellular decomposition of S and calculate its homology groups.

2.6. Consider the following two–dimensional cellular complex C. Its one skeleton C_1 is the circle identified with the unit circle in the complex plane (It can be viewed as a one–cell attached to a zero–cell). It has two two–dimensional cells c_1 and c_2 The characteristic maps of c_1 and c_2 correspondinly have the form $\mathbb{D} \to C$ where $\mathbb{D}$ is the unit disc in the complex plane. Their restrictions to the boundary $\partial\mathbb{D} \to C_1$ are the maps of the unit circle of the form $z \to z^4$ and $z \to z^5$ correspondingly.

Calculate homology groups of C.