ANALYSIS PH.D QUALIFYING EXAMINATION

May 11, 1998

Complete solutions of two problems from each of the sections below are sufficient for passing this examination. Partial credits will be given.

You may use standard results from the 501–502 sequence. If you are not sure whether a certain result is considered standard consult the faculty member proctoring the exam.

References to results beyond the syllabus should be provided with convincing explanations.

Section 1. Real Analysis

1. Let f be a Lebesgue integrable function on $[0,1]$ and $0 < c < 1$. Assume that $\int_E f(t)\,dt = 0$ for every measurable set $E \subset [0,1]$ with $\lambda(E) = c$. Prove that f must vanish almost everywhere.

2. Consider the function $f : [0,1) \times [0,1) \to \mathbb{Z} \cup \{\infty\}$ defined as follows. For $x, y \in [0,1)$, let $x = 0.i_1(x)\ldots i_n(x)\ldots$ and $y = 0.i_1(y)\ldots i_n(y)\ldots$ be standard decimal expansions of x and y with infinitely many digits other than 9. Then $f(x,y)$ equals the smallest k for which $i_k(x) = i_k(y)$ or $f(x,y) = \infty$ if no such k exists. Show that f is Lebesgue measurable.

3. Show that if $f \in L^3([-1,1])$ then the integral

$$\int_{-1}^{1} \frac{f(x)}{\sqrt{|x|}}\,dx$$

is finite.

4. Let $\{f_n\}$ be a sequence of measurable functions on a finite measure space $(X, \mathcal{M}, \mu)$. Suppose that

$$\sum_{n=1}^{\infty} \mu(\{x \in X \,:\, |f_n(x)| > 1\}) < \infty.$$

Prove that

$$-1 \le \liminf_{n\to\infty} f_n(x) \le \limsup_{n\to\infty} f_n(x) \le 1$$

for μ-almost every $x \in X$.

Section 2. Functional Analysis

1. Consider a sequence $a = (a_i)_{i=1}^{\infty} \in l_2$ for which $a_i \ne 0$ for all $i \ge 1$. Prove that there is a sequence $b = (b_i)_{i=1}^{\infty} \in l_1$ such that the sequence

$$\left(\frac{b_i}{a_i}\right)_{i=1}^{\infty}$$

is not in l_2.

Hint: You may consider the linear map $T_a(c) = (a_i c_i)_{i=1}^{\infty}$, where $c = (c_i)_{i=1}^{\infty} \in l_2$.

2. Let A_n be a sequence of continuous linear operators on a Hilbert space H. Assume that the sequence $A_n(x)$ is convergent for every $x \in H$. Prove that the sequence $A_n B$ is convergent in the norm for any compact linear operator B.

3. Let φ be a Lebesgue measurable function on $[0,1]$ which takes on only two values -1 and 1 each on a set of positive measure. Consider also a linear bounded operator on $L^2([0,1],\lambda)$ (where λ is the Lebesgue measure) given as

$$A_\varphi(f) = \varphi f, \quad f \in L^2([0,1],\lambda).$$

Find the spectrum of A_φ.

4. Consider the vector space of all real polynomials with the norm

$$\|p\| = \sup_{x\in[-1,1]} |p(x)|.$$

Show that the functional $\varphi(p) = \frac{dp}{dx}(0)$ is not continuous in this norm.

Section 3. Complex Analysis

1. Let $\mathcal{S}$ be a family of Möbius transformations. Suppose that there are two points $x, y \in \mathbb{C}_\infty$ (where $\mathbb{C}_\infty = \mathbb{C} \cup \{\infty\}$) are such that $f(x) = x$ and $f(y) = y$ for any $f \in \mathcal{S}$. Prove that $fg = gf$ for any two $f, g \in \mathcal{S}$.

2. Show that

$$\int_0^\infty \frac{\cos x \; dx}{x^2+1} = \frac{\pi}{2e}.$$

3. For every $n > 0$, consider the function

$$f_n(z) = \sum_{k=0}^{n} \frac{(\sin z)^k}{k!}.$$

Fix a number $R > 0$. Prove that there exists $N > 0$ such that for every $n \geq N$, $f_n(z) \neq 0$ on the disk $D(0,R)$ (centered at 0 of radius R).

4. Let f be an entire function which satisfies

$$|f(z)| < \exp(|z|)$$

for every $z \in \mathbb{C}$. Show that there exists $K > 0$ such that for every $z \in \mathbb{C}$

$$|f'(z)| < K \exp(|z|).$$