

Qualifying Exam in Analysis

Do two out of four problems in each of the three sections of the exam.

1. REAL ANALYSIS

1. Let μ and ν be two finite positive measures on a σ -algebra \mathfrak{M} . Assume that ν is absolutely continuous with respect to μ in the sense that $\nu(E) = 0$ for every $E \in \mathfrak{M}$ for which $\mu(E) = 0$. Prove that for every $\epsilon > 0$, there exists $\delta > 0$, for which $\mu(A) < \delta$ implies that $\nu(A) < \epsilon$.

2. Let $g : \mathbb{R} \rightarrow \mathbb{R}$ be continuous and 2π -periodic, and let $f(x) = \int_0^{2\pi} g(x-y)g(y) dy$ denote the convolution of g with itself. Prove that if f is C^∞ then g is C^∞ .

3. Let $\phi(x)$ be a continuous, strictly increasing, real-valued function on $[0, \infty)$, with $\phi(0) = 0$, and $\lim_{x \rightarrow +\infty} \phi(x) = +\infty$. Define $\Phi(x) = \int_0^x \phi(t) dt$ and $\Psi(y) = \int_0^y \phi^{-1}(t) dt$. Show that

$$xy \leq \Phi(x) + \Psi(y) \quad \text{for all } x, y \geq 0.$$

4. Let I denote the interval $(0, 1)$ and let f be a real-valued function on I whose derivative exists everywhere and is bounded on I . Prove that $f(E)$ has measure zero for every set E of I of measure zero.

2. COMPLEX ANALYSIS

5. Let H denote the set of all analytic functions mapping the open unit disk D into itself. Find

$$\sup_{f \in H} |f''(0)| \quad \text{and} \quad \sup_{\substack{f \in H \\ z \in D}} |f''(z)|.$$

6. Let f be an analytic function on a region G and suppose that there exists a differentiable function $g : \mathbb{R} \rightarrow \mathbb{R}$ such that $\operatorname{Re} f(z) = g(\operatorname{Im} f(z))$ for all $z \in G$. Prove that f is constant.

7. Use contour integration in the complex plane to evaluate $\int_{-\infty}^{+\infty} e^{-x^2} \cos(x) dx$.

8. Let $S = \{z \in \mathbb{C} \mid 0 \leq \operatorname{Im} z \leq 1\}$. Suppose that f is holomorphic on an open neighborhood of S , maps S into itself, and satisfies $f(x+i) = f(x)$ for all $x \in \mathbb{R}$. Prove that f is constant.

3. FUNCTIONAL ANALYSIS

9. Let $e_i, i = 1, 2, \dots$ be the usual orthonormal basis for ℓ_2 , and let $T : \ell_2 \rightarrow \ell_2$ be such that $\sum_{i,j=1}^{\infty} |\langle e_i, T e_j \rangle|^2 < +\infty$. Suppose that the restriction of T to some closed subspace S of ℓ_2 is the identity. Prove that S is finite dimensional.
10. A subset S of a Banach space X is called *weakly bounded* if for all $f \in X^*$ the set $f(S)$ is bounded. Prove that a subset is weakly bounded if and only if it is bounded in norm.
11. Let K be a convex subset of a Hilbert space, and x an element of the space. Show that there cannot exist two distinct elements y, z of K such that $\|x - y\| = \|x - z\| = \text{dist}(x, K)$. Show by example that this can happen in a Banach space.
12. Suppose that M and N are closed subspaces of a Banach space X such that $M + N = X$ and $M \cap N = 0$. Prove that X/M is isomorphic to N .