Qualifying Exam in Analysis

May, 1996

This is a controlled examination. No books or notes are allowed and any of communication during the examination is strictly forbidden. Violation of these rules will result in failing the exam and further disciplinary action.

Two problems in each of the three sections solved flawlessly is considered a perfect score and a guaranteed pass. In other cases, doing more than two problems in any section may help, while getting a low score in even one section may hurt.

If you refer to a theorem without proving it, you should state it fully and indicate where the proof can be found. The source should be easily verifiable. An exception from this is when you use a very well-known theorem with an unambiguous name. For example if you say "the Riesz-Fischer Theorem", no statement or reference to a source is needed. On the other hand a "Riesz Theorem" may get you in trouble, since there is more than one.

Functional Analysis

Problem 1. Let $\mathbf{R}^\infty$ be the vector space of real sequences $(x_i)_{i=1}^\infty$ with a distance

$$d((x_i),(y_i)) := \sum_{i=1}^\infty 2^{-i} \frac{|x_i - y_i|}{1+|x_i-y_i|}.$$

Prove that $\mathbf{R}^\infty$ with the topology introduced by this distance is *not* a normable topological vector space.

Problem 2. Let X be a Banach space of infinite dimansion. Prove that X contains a linear subspace which is not closed.

Problem 3. Let H be a Hilbert space with $T : H \to H$ a compact linear operator. Suppose that $\lambda \neq 0$ is a complex number. Prove that the following statements are equivalent:

- There is $0 \neq x \in H$ so that for every $y \in H$
$$(Tx, y) = \lambda(x, y).$$
- There is $0 \neq y \in H$ so that for every $x \in H$
$$(Tx, y) = \lambda(x, y).$$

Problem 4. Let X be a Banach space. Let X_w be X with its weak topology. Prove that X_w is of the second category in itself if and only if the dimension of X is finite.

Reminder: A topological space Y is of second category in itself if and only if Y cannot be expressed as a countable union of closed subsets with empty interiors.

Real Analysis

Problem 1. Let g and h be continuous, complex-valued functions on $[0,1]$. Suppose that h is absolutely continuous and denote with G an anti-derivative of g. Prove that for every $x \in [0,1]$:

$$\int_0^x g(t)h(t)dt = G(x)h(x) - G(0)h(0) - \int_0^x G(t)h'(t)dt.$$

Problem 2. All functions mentioned in this problem are supposed to be real-valued. Assume that $f_n \in C([-1,1])$ for $n = 1, \ldots$. Suppose that for every $g \in L^2([-1,1])$

$$\lim_{n\to\infty} \int_{-1}^1 f_n(x)g(x)dx = 0.$$

Does it follow that f_n tend to 0 in the sup-norm on $C([-1,1])$?

Problem 3. Consider the Banach space of complex regular Borel measures on $[0,1]$ with the norm given by the total variation. Consider the set of measures which are absolutely continuous with respect to the Lebesgue measure. Is this set closed?

Problem 4. Let f be defined on $[0,1]$, complex-valued, and in L^1 with respect to the Lebesgue measure. Suppose that for every continuous function ϕ on $[0,1]$ with values in non-negative real numbers

$$0 \leq \int_0^1 \phi(x)f(x)dx \leq \sup\{\phi(x) : x \in [0,1]\}.$$

Prove that f takes real and non-negative values almost everywhere in the sense of the Lebesgue measure.

Complex Analysis

Problem 1. Prove that there is a function U, harmonic and bounded in $D(0,1)$ for which there is no function V harmonic and bounded in $D(0,1)$ so that $U + iV$ is holomorphic.

Problem 2. Let ϕ be an entire function and suppose that

$$|\phi(z)| \leq 1 + \sqrt{|z|}$$

for every $z \in \mathbf{C}$. Prove that ϕ is constant.

Problem 3. Consider a function ϕ holomorphic in $D(0,1)$ mapping into $D(0,1)$. Prove that if $\alpha \in D(0,1)$ and $\phi(\alpha) = 0$, then $|\alpha| \geq |\phi(0)|$.

Problem 4. Calculate

$$\int_{-\pi}^{\pi} \frac{\exp(e^{2it})(e^{it}-1/2)}{1-1/2e^{it}} dt.$$