ANALYSIS QUALIFYING EXAMINATION

MAY 12, 1995

There are 4 problems from Complex Analysis, 4 problems from Real Analysis and 4 problems from Functional Analysis. A perfect score will be awarded for doing 2 problems from each section. Partial credit will be awarded.

Complex Analysis

1. Let $f : \mathbb{C} \to \mathbb{C}$ be an analytic function which is one-to-one and onto. Show that there exist $a, b \in \mathbb{C}$ such that $f(z) = az + b$ for all $z \in \mathbb{C}$.

2. Let $f_1, f_2, \ldots$ be a sequence of analytic functions on a connected open set G, and assume that each f_n is one-to-one. Suppose that f_n converges, uniformly on compact sets, to an analytic function f. Prove that f is either one-to-one or constant.

3. Find the residue of the function $\cos(z)^2 / \sin(z)^2$ at the origin.

4. Let f be analytic on a region containing the closed unit disk $\overline{D}$. Let w_1 and w_2 be points in the open unit disk D. Evaluate the integral

$$\frac{1}{2\pi i} \int_{\partial D} \frac{f(z)}{(z - w_1)(z - w_2)} dz$$

in terms of $w_1, f(w_1)$ and $f(w_2)$.

Real Analysis

1. Prove that any countable set of real numbers is the set of points of discontinuity of some monotone function.

2. Prove that if f and g are (real-valued) measurable functions then so are the functions $f + g$ and fg.

3. A real number a is a *condensation point* of a set A if every neighborhood of a contains uncountably many elements of A. Let A^* denote the set of all condensation points of A. Show that if F is an uncountable closed set then (a) F^* is perfect, (b) $F - F^*$ is countable, and (c) if $P \subseteq F$ is perfect then $P \subseteq F^*$.

4. Let μ be a σ-finite measure and let $\mathcal{A} = \{A_i : i \in I\}$ be a family of sets such that $\mu(A_i) > 0$ for every $i \in I$ and $\mu(A_i \cap A_j) = 0$ whenever $i \neq j$. Show that $\mathcal{A}$ is at most countable.

Typeset by $\mathcal{A}_{\mathcal{M}}\mathcal{S}$-TEX

Functional Analysis

1. Let V be a real Banach space and let $f : \mathbb{R} \to V$ be a function such that $\eta \circ f : \mathbb{R} \to \mathbb{R}$ is a differential function, for every continuous linear functional $\eta : V \to \mathbb{R}$. Prove that for every $x \in \mathbb{R}$, $\lim_{t \to 0}(f(x+t) - f(x))/t$ exists in the norm topology of V.

2. Give an example of a Banach space V and a sequence of closed and bounded subsets B_i in V so that $B_1 \supset B_2 \supset \cdots$ and $\cap_{i=1}^{\infty} B_i = \emptyset$. On the other hand, prove that if B_i are also required to be closed balls, no such example exists.

3. Let A and B be a compact linear operators from a Banach space V into itself. Prove that $A + B$ is compact as well.

4. Prove that every vector subspace of a Banach space which is closed in the norm topology is also closed in the weak topology.