ANALYSIS QUALIFYING EXAMINATION

MAY 14, 1993

There are 4 problems from Complex Analysis, 4 problems from Real Analysis and 4 problems from Functional Analysis. A perfect score will be awarded for doing 2 problems from each section. Partial credit will be awarded, but given a choice between writing something incorrect and writing nothing at all, your grade will suffer less from writing nothing at all.

Complex Analysis

1. Prove that

$$\int_0^\infty \frac{\log^2 x}{1+x^2}\,dx = \frac{\pi^3}{8} \qquad \text{and} \qquad \int_0^\infty \frac{\log x}{1+x^2}\,dx = 0.$$

2. Show that if f is an entire, periodic function, then it has a fixed point.

3. Let R denote a closed domain bounded by two circles C_1 and C_2 such that C_2 lies in the interior of the disk bounded by C_1. Prove that there is some $0 < r < 1$ and a linear fractional transformation that maps R onto the annulus

$$A_r = \{z \in \mathbb{C} \mid r \le |z| \le 1\}.$$

4. Show that if f is a holomorphic map of a bounded, simply connected domain into itself that has two fixed points, then $f(z) = z$.

Real Analysis

1. Let λ denote Lebesgue measure on $[0,1]$ and suppose $p \geq 1$. Show that if $\{f_n\}$ is a sequence of λ–measurable functions on $[0,1]$ such that

   (i) $|f_n| < g$ for some $g \in L_p([0,1],\lambda)$ and

   (ii) $f_n \overset{\lambda}{\to} f$,

then $f_n \to f$ in $L_p([0,1],\lambda)$. (Here $f_n \overset{\lambda}{\to} f$ means that f_n converges to f in measure).

2. Suppose μ is a non–atomic measure on the Borel subsets of $[0,1]$ such that μ and Lebesgue measure are mutually singular. Show that if we write $f(x) = \mu([0,x])$, then

$$\limsup_{h\to 0} \frac{f(x+h)-f(x)}{h} = \infty$$

except possibly on a set of μ–measure 0.

3. Suppose (X,μ) and (Y,ν) are σ–finite measure spaces and ρ is a measure on $X\times Y$ that is absolutely continuous with respect to $\mu\times\nu$. Prove that for μ–almost every $x \in X$ there exists a measure ν_x on Y that is absolutely continuous with respect to ν such that for every $f \in L_1(X\times Y,\rho)$ the following statements hold.

   (1) For μ–almost every $x \in X$, $f(x,\cdot) \in L_1(Y,\nu_x)$ and, as a function on X,

$$\int_Y f(x,y)d\nu_x \in L_1(X,\mu),$$

   (2) $\int_{X\times Y} f(x,y)d\rho = \int_X (\int_Y f(x,y)d\nu_x)d\mu$.

4. Let λ denote Lebesgue measure on $[0,1]$ and write $\mathcal{B}$ for the σ–algebra of Lebesgue measurable sets on $[0,1]$. Suppose A is a set in $[0,1]$ such that $\lambda_*(A) = 0$, $\lambda^*(A) = 1$. (Here λ_* and λ^* denote inner measure and outer measure, respectively).

   (1) Construct an extension $\tilde{\lambda}$ of λ to the minimal σ–algebra $\mathcal{B}_A$ that contains $\mathcal{B}$ and A such that $\tilde{\lambda}(A) = 1/2$.

   (2) Construct an isomorphism mod 0 between the σ-algebras $(\mathcal{B}_A, \tilde{\lambda})$ and $(\mathcal{B}, \lambda)$.

(Recall that if $\mathcal{Z}(B)$ stands for the collection of sets of measure 0 in $\mathcal{B}$, $\mathcal{B}/\mathcal{Z}(B)$ becomes a metric space with respect to the distance $\rho(A,B) = \lambda(A\Delta B)$, and an isomorphism mod 0 is an isometry of $\mathcal{B}_A/\mathcal{Z}(B_A)$ onto $\mathcal{B}/\mathcal{Z}(B)$ that preserves complements and finite unions.)

Functional Analysis

1. Let $\mathcal{H}$ denote an infinite dimensional, separable Hilbert space and write $S = \{x \in \mathcal{H} \mid \|x\| = 1\}$. Find the closure of S in the weak topology.

2. Suppose $\mathcal{H}$ is an infinite dimensional, separable Hilbert space and $A : \mathcal{H} \to \mathcal{H}$ a bounded normal operator, i.e. $AA^* = A^*A$. Prove that if λ and μ are complex numbers such that $\lambda \neq \mu$ and f_n, g_n are sequences in $\mathcal{H}$ such that $\|f_n\| = \|g_n\| = 1$ and

$$\|Af_n - \lambda f_n\| \to 0, \quad \|Ag_n - \mu g_n\| \to 0$$

as $n \to \infty$, then $\langle f_n, g_n \rangle \to 0$ as $n \to \infty$.

3. Let $\mathbb{R}^\infty$ denote the linear space of all sequences of real numbers with the product topology. Prove that this topology cannot be generated by a norm.

4. Suppose $\mathcal{X}$ is a normed linear space and S is a subset of $\mathcal{X}$. Show that if

$$\sup_{s \in S}\{|f(s)|\} < \infty$$

for every $f \in \mathcal{X}^*$, then S is bounded in norm.