ANALYSIS QUALIFYING EXAMINATION

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There are 4 problems from Complex Analysis and 8 problems from Real and Functional Analysis. Work at least 2 problems from the Complex Analysis section and at least 4 problems from the Real and Functional Analysis section. Take care in what you write. Incorrect statements will detract from your score.

Complex Analysis

1. Suppose f is analytic in $\{z : |z| < 1\}$ and that it has a zero of order $k \geq 2$ at $z = 0$. What type of singularity does $\frac{f''(z)}{f(z)}$ have at $z = 0$? If it is isolated, then determine the residue.

2. Use the residue theorem to evaluate the integral $\int_0^{2\pi} \sin^{2n} \theta \, d\theta$.

3. Let $f(z)$ be an analytic function in $\{z : |z| < 1\}$ such that $\text{Re} f(z) > 0$. Show that

$$|f'(0)| \leq 2 \text{ Re } (f(0)).$$

4. Find the number of roots of the equation $6z^4 + z^3 - 2z^2 + z - 1 = 0$ in the disc $|z| < 1$

Real and Functional Analysis

1. State the Monotone Convergence Theorem and use it to prove Fatou's Lemma. Recall that Fatou's Lemma says the if $f_n : X \to [0, \infty]$ is measurable, for each positive integer n, then

$$\int_X \left(\liminf_{n \to \infty} f_n \right) d\mu \leq \liminf_{n \to \infty} \int_X f_n d\mu.$$

2. Prove that if X and Y are topological spaces, $f : X \to Y$ is continuous, and $B \in \mathcal{B}(Y)$, the σ-algebra of Borel sets in Y, then $f^{-1}(B) \in \mathcal{B}(X)$.

3. Suppose $1 \leq p \leq \infty$, $f \in L^1(\mathbb{R})$, and $g \in L^p(\mathbb{R})$. Prove that $(f * g)(x) = \int_{-\infty}^{\infty} f(x-y)g(y)dm(y) \in L^p(\mathbb{R})$, and that

$$\|f * g\|_p \leq \|f\|_1 \|g\|_p.$$

4. Let Z be a subspace of a normed linear space X and suppose that $\mathbf{y}$ is an element of X such that the distance from Z to $\mathbf{y}$ is d. (Recall that $dist(\mathbf{y}, Z) =$

$\inf\{\|\mathbf{y}-\mathbf{z}\| | \mathbf{z} \in Z\}$.) Prove that there exists $\Lambda \in X^*$ (i.e. a continuous linear functional on X), such that $\|\Lambda\| \leq 1, \Lambda(\mathbf{y}) = d$, and $\Lambda(\mathbf{z}) = 0$, for every $\mathbf{z} \in Z$.

5. Let $\{f_n\} \subset L^1([0,1])$ be a sequence of real Lebesgue integral functions on $[0,1]$, and let $f \in L^1([0,1])$ be such that $\lim_{n\to\infty} f_n(x) = f(x)$ a.e. with respect to Lebesgue measure. Prove that if $\lim_{n\to\infty} \int_0^1 |f_n| dm = \int_0^1 |f| dm$, then

$$\lim_{n\to\infty} \int_0^1 |f_n - f| dm = 0.$$

(Hint: You may wish to use the fact that for every $\epsilon > 0$, there exists $\delta > 0$, such that if E is a measurable set with $\mu(E) < \delta$, then $\int_E |f| d\mu < \epsilon$.)

6. Let $X = \ell^2(\mathbb{N})$. Recall that a sequence $\{\mathbf{x}^{(n)}\}$ of points in X converges **weakly** to $\mathbf{x} \in X$ if for every $\Lambda \in X^*$,

$$\lim_{n\to\infty} \Lambda(\mathbf{x}^{(n)}) = \Lambda(\mathbf{x}).$$

Prove that if $\mathbf{y} \in \ell^2(\mathbb{N})$ with $\|\mathbf{y}\|_2 \leq 1$, then there exists $\{\mathbf{x}^{(n)}\} \subset \ell^2(\mathbb{N})$, with $\|\mathbf{x}^{(n)}\|_2 = 1$ for all n, such that $\{x^{(n)}\}$ converges weakly to $\mathbf{y}$; i.e., the closure of the unit sphere (with respect to the weak topology) is the unit ball.

7. Let H be a complex Hilbert space and M a linear functional on H. Define $ker(M) = \{\mathbf{y} \in H | M(\mathbf{y}) = 0\}$. Prove that if M is continuous, $ker(M)$ is a closed subspace of H and if M is discontinuous then $ker(M)$ is dense in H.

8. Let X and Y be a Banach spaces, and suppose $B : X \times Y \to \mathbb{C}$ is a bilinear functional. Suppose further that for every fixed $\mathbf{x} \in X$, $B(x, \cdot)$ is continuous linear functional on y and for every fixed $\mathbf{y} \in B(\cdot, \mathbf{y})$ is a continuous linear functional on X. Prove that if $\lim_{n\to\infty} \mathbf{x}_n = 0$, and $\lim_{n\to\infty} \mathbf{y}_n = 0$, for sequences $\{\mathbf{x}_n\}$ and $\{\mathbf{y}_n\}$ respectively, then $\lim_{n\to\infty} B(\mathbf{x}_n, \mathbf{y}_n) = 0$. (Another way of saying this is that B is jointly continuous in $\mathbf{x}$ and $\mathbf{y}$.)