ANALYSIS QUALIFYING EXAMINATION

MAY 17, 1991

There are 4 problems from Complex Analysis and 8 problems from Real and Functional Analysis. Work at least 2 problems from the Complex Analysis section and at least 4 problems from the Real and Functional Analysis section. Take care in what you write. Incorrect statements will detract from your score.

Complex Analysis

1. In the following statements f denotes a **nonconstant** function that is analytic in the punctured disk $\Omega = \{0 < |z| < 2\}$. Indicate which statements are true and which are false. Give a brief reason for your answers.

a. f may have zeros at $z_n = 1/n, n = 1, 2 \ldots$.

b. If

$$\lim_{x \to 0} f(x) = 0, \text{ then } \lim_{z \to 0} \frac{1}{f(z)} = \infty,$$

where $z = x + iy$.

c. If $|f(z)| = 1$ for $|z| = 1$ and $|f(z)| \geq 1$ for $0 < |z| < 1$, then

$$\{f(z) : 0 < |z| < 1\}$$

is not bounded.

c. If $f'(z) \neq 0$ for $z \in \Omega$, then f is $1 - 1$.

e. If

$$\int_{|z|=1} f(z) dz = 0,$$

then f has a removable singularity at $z = 0$.

2. Find a conformal mapping of the region outside the circle $|z - 1| = 1$ and inside the circle $|z| = 2$ onto the unit disk $|w| < 1$. (Hint: first map an appropriate point to ∞.)

3. Evaluate the integral

$$\int_0^\infty \frac{(\log x)^2}{1 + x^2} dx.$$

4. Define what it means for an analytic function to have a removable singularity at a point a. Prove that if f has an isolated singularity at a, then the point $z = a$ is a removable singularity iff

$$\lim_{z \to a} (z - a) f(z) = 0.$$

Real and Functional Analysis

1. Consider the following statement:

"If $\{f_n\}$ is a sequence of nonnegative Lebesgue measurable functions defined on $\mathbb{R}$ such that

i. $f_n(x) \geq f_{n+1}(x)$ for all $x \in \mathbb{R}$ and all n; and

ii. the sequence $\{f_n\}$ converges pointwise to a function f on $\mathbb{R}$, then

$$\lim_{n\to\infty} \int_{\mathbb{R}} f_n \, dm = \int_{\mathbb{R}} f \, dm,$$

where m denotes Lebesgue measure."

a) Give a counter-example that shows that the statement is false in general.

b) Prove that with an additional hypothesis, the statement is true.

2. Suppose X is a topological space and μ is a σ-finite measure on the Borel subsets of X such that if E is a Borel set and $\mu(E) < \infty$, then for each $\epsilon > 0$ there is an open set V containing E such that $\mu(V \backslash E) < \epsilon$

a) Prove that if E is any Borel set then for each $\epsilon > 0$ there is an open set V containing E such that $\mu(V \backslash E) < \epsilon$

b) Prove that if E is any Borel set then there is an F_σ set A and a G_δ set B such that $A \subset E \subset B$ and $\mu(B \backslash A) = 0$.

3. State Hölder's inequality and Minkowski's inequality. Derive the latter from the former.

4. Let $C_0(\mathbb{R})$ denote the continuous complex-valued functions on $\mathbb{R}$ that vanish at ∞. Suppose that $\{g_n : n \in \mathbb{N}\}$ is a sequence in $C_0(\mathbb{R})$ with the following properties:

(i) $$g_n(t) \geq 0 \text{ for } t \in \mathbb{R}.$$

(ii) $$\int_{-\infty}^{\infty} g_n(t)dt = 1.$$

(iii) $$\int_{|t|>1/n} g_n(t)dt < \frac{1}{n},$$

for every $n \in \mathbb{N}$.

Prove that if $f \in C_0(\mathbb{R})$ and we associate functions h_n to f defined by

$$h_n(t) = \int_{-\infty}^{\infty} f(t-s)g_n(s)ds \qquad (k = 1, 2, 3, \dots),$$

then $h_n \in C_0(\mathbb{R})$ and $\lim_{n\to\infty} \|f - h_n\|_\infty = 0$.

5. Suppose K is a compact Hausdorff space and ϕ is a continuous linear functional on $C(K)$ such that

$$\phi(1) = \|\phi\| = 1,$$

where 1 is the constant function. Prove that ϕ is a positive linear functional.

6. Prove that if T is a linear transformation of a Hilbert space H into itself such that

$$(Tx, y) = (x, Ty) \quad \text{for all } x, y \in H,$$

then T is continuous.

7. Suppose $\{X, \mathcal{M}, \mu\}$ is a σ-finite measure space, $1 \le p < \infty$ and ϕ is a continuous linear functional on $L^p(\mu)$.

a) Prove that if $1 < p$ then there is an element $f \in L^p(\mu)$ such that $\|f\|_p = 1$ and $\phi(f) = \|\phi\|$.

b) Prove that a) need not be true when $p = 1$.

8. Let T denote an operator in $\mathcal{B}(H)$.

(1) Prove that if e is an eigenvector for $|T|$ and $x \in H$ is orthogonal to e, then $(Te, Tx) = 0$.

(2) Prove that if T is compact then there is an orthonormal basis $\{e_n\}$ for H and an orthogonal set $\{f_n\}$ in H such that $Te_n = f_n$, $n = 1, 2, \ldots$.