Analysis Qualifying Examination

May 14, 2007

Instructions. Do two problems from each section. Only your top two problem scores from each section will be recorded.

Section 1 - Real Analysis.

Problem 1. State the Monotone Convergence theorem and the Dominated Convergence theorem, then prove the Dominated Convergence theorem using the Monotone Convergence theorem.

Problem 2. Define a function $f\colon \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} x^{-1/2} & 0 < x < 1 \\ 0 & \text{otherwise.} \end{cases}$$

Let $\{r_n\}$ be an enumeration of the rationals, and set

$$g(x) = \sum_{n=1}^{+\infty} 2^{-n} f(x - r_n).$$

Show the following:

(a) g is discontinuous at every point and unbounded on every interval;

(b) $g \in L^1(\mathbb{R})$, and so in particular $g < +\infty$ a.e.;

(c) g^2 is not integrable on any proper interval.

Problem 3. Let $f : X \to [0, +\infty]$ be a measurable function on a measure space. Show that there exists a sequence $\{A_j\}$ of measurable sets such that

$$f = \sum_{j \geq 1} \frac{1}{j} \chi_{A_j}.$$

Problem 4. Let μ be a positive, finite Borel measure on $\mathbb{R}$ such that for every Borel set A, the function $y \mapsto \mu(A+y)$ is continuous in y.

(a) Let χ_A be the characteristic function of a Borel set A. Let m be Lebesgue measure on $\mathbb{R}$. Apply Fubini's theorem to the function $\chi_A(x-y)$ to conclude that if $m(A)=0$, then $\mu(A)=0$

(b) Show that there exists a function $f \in L^1(\mathbb{R}, m)$ such that

$$\mu(A) = \int_A f(y)\, dm(y)$$

for every Borel set A.

Section 2 - Functional Analysis.

Problem 5. Let $\|\cdot\|$ be a norm on $C[a,b]$ such that $(C[a,b], \|\cdot\|)$ is a Banach space and such that whenever $\|f_n - f\| \to 0$, then $f_n(t) \to f(t)$ for all $t \in [a,b]$. Prove that the norm $\|\cdot\|$ is equivalent to the supremum norm $\|\cdot\|_\infty$.

Problem 6. Let $p \in (1,\infty)$. Prove that a linear operator $T\colon L^p([0,1]) \to L^p([0,1])$ is compact if and only if $\|Tf_n\|_p \to 0$ for every sequence $\{f_n\} \in L^p([0,1])$ that converges weakly to 0.

Problem 7. Let H be an inner product space. Prove that if the Riesz representation theorem holds for H, then H is a Hilbert space.

Problem 8. A linear operator T on a Hilbert space H is said to be *normal* if $TT^* = T^*T$.

(a) Let $\{\lambda_n\}$ be a bounded sequence of complex numbers and let $T : l_2 \to l_2$ be defined by $T\{x_n\} = \{\lambda_n x_n\}$. Prove that T is normal and find its spectrum.

(b) Use (a) to conclude that if H is an infinite-dimensional separable Hilbert space, then any compact subset of $\mathbb{C}$ is the spectrum of a normal operator on H.

Section 3 - Complex Analysis.

Problem 9. Prove the following part of the Schwarz lemma: if f is an analytic function from the unit disk in $\mathbb{C}$ to itself, and if $f(0)=0$, then $|f(z)| \leq |z|$ for every element z of the unit disk.

Problem 10. Find a fractional linear transformation that maps the half-disk

$$\{z \in \mathbb{C} : |z| < 1 \text{ and } \operatorname{Im}(z) > 0\}$$

bijectively onto the quadrant

$$\{z \in \mathbb{C} : \operatorname{Re}(z) > 0 \text{ and } \operatorname{Im}(z) > 0\}.$$

(Prove that your map has all the properties required of it.)

Problem 11. Let f be analytic on an open set containing the closed unit disk. Let γ be the boundary of the unit disk (considered as a smooth closed curve with its usual counter-clockwise orientation) and let w_1 and w_2 be distinct points in the open unit disk. Evaluate the integral

$$\frac{1}{2\pi i}\int_\gamma \frac{f(z)}{(z-w_1)(z-w_2)}\,dz.$$

Problem 12. Show that if $a > 1$, then

$$\int_0^\pi \frac{dt}{a+\cos t} = \frac{\pi}{\sqrt{a^2-1}}.$$