Ph.D. QUALIFYING EXAMINATION IN ANALYSIS

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To obtain a perfect score, you should give complete solutions to two problems in each section.

Section 1 - Measure Theory

1. Let (X, μ) be a finite measure space and $f \in L^1(X, d\mu)$. Show that, for any $\epsilon > 0$, there exists $\delta > 0$ such that $\left| \int_E f(x) d\mu(x) \right| < \epsilon$ for any measurable set $E \subset X$ such that $\mu(E) < \delta$.

2. Let (X, μ) be a finite measure space and $f : X \to [0, \infty]$ be a measurable function. Find the limit
$$\lim_{n\to\infty} n \int_X \left[(1 + f(x)/n)^{1/3} - 1 \right] d\mu(x).$$

3. Denote by δ_a the Dirac measure on $[0, 1]$ concentrated at a (so $\delta_a(E) = 1$ if $a \in E$, $\delta_a(E) = 0$ otherwise). Show that $\int_0^1 f(x) d\delta_a(x) = f(a)$. Show that $\mu_n = (\sum_{j=1}^n \delta_{j/n})/n$ is a sequence of positive measures on $[0, 1]$ of norm 1. Prove that μ_n is weakly convergent to a measure ν but $\|\nu - \mu_n\| = 2$.

4. The convolution of two functions $f, g \in L^1(\mathbb{R})$ is defined by:
$$f * g(x) = \int_{\mathbb{R}} f(x - y)\, g(y)\, dy.$$
   (a) Prove that the integral defining $f * g$ exists almost everywhere in x (with respect to the Lebesgue measure).
   (b) Prove that $\|f * g\|_{L^1} \leq \|f\|_{L^1} \|g\|_{L^1}$.
   (c) Prove that a closed subspace $X \subset L^1(\mathbb{R})$ is translation-invariant (that is, $f \in X \Rightarrow f_a \in X$, where $f_a(x) = f(x + a)$) if and only if X is an ideal, that is, $f \in X$, $g \in L^1(\mathbb{R}) \Rightarrow f * g \in X$.

Section 2 - Functional Analysis

1. Disprove by counter-example the following statement:

   if T is a bounded, self-adjoint operator on a complex Hilbert space $\mathcal{H}$ and $\|x\| \leq \|Tx\| \leq 2\|x\|$, $\forall x \in \mathcal{H}$, then there is a $\lambda \in \mathbb{R}$, and $y \in \mathcal{H}$, $y \neq 0$, such that $Ty = \lambda y$.

2. Let X be a Banach space and $B : X \times X \to \mathbb{R}$ be a bilinear form such that:

   i) For all $x \in X$ fixed, the map $y \to B(x, y)$ is continuous.

   ii) For all $y \in X$ fixed, the map $x \to B(x, y)$ is continuous.

   Show then that there exists a linear, bounded operator $T : X \to X^*$ such that $B(x, y) = \langle Tx, y\rangle$, where $\langle \cdot, \cdot\rangle : X^* \times X \to \mathbb{R}$ is the usual duality form: $\langle f, x\rangle = f(x)$. Conclude that there exists $C > 0$ such that $|B(x, y)| \leq C\,\|x\|\|y\|$.

3. Show that there is a bounded linear functional λ on $L^\infty([-1, 1])$ such that $\lambda(f) = f(0)$ for each $f \in C([-1, 1])$. (The space $L^\infty([-1, 1])$ is defined using the Lebesgue measure.)

4. For $f \in L^2([0, 1])$, define a linear operator T by

$$Tf(x) = \int_0^1 \frac{1 + x^2}{1 + y^2}\, e^{xy}\, f(y)\, dy.$$

   (a) Prove T is a bounded operator on $L^2([0, 1])$ and compute its adjoint.

   (b) Show there exists a bounded, invertible, *multiplication* operator $U : L^2 \to L^2$, $Uf = \phi f$ for some function ϕ, such that $U^{-1}TU$ is self-adjoint. Show that $U^{-1}TU$ and T have the same spectrum, which is moreover contained in $[-e, e]$.

Section 3 - Complex Analysis

1. Find a bijective, conformal map of the half-strip $\{z = x + i\,y \;\mid\; x > 0, 0 < y < 1\}$ to the unit disk.

2. Compute, using residues:

$$\int_{-\infty}^{+\infty} \frac{e^{iat}}{t-i}\,dt, \qquad a \in \mathbb{R} \setminus \{0\}.$$

   Justify carefully your work.

3. Let f be holomorphic in the unit disk, such that $|f(z)| \leq 1$, for $|z| < 1$, and $f(0) = 0$, $f'(0) = 0$. Prove that $|f''(0)| \leq 2$.

4. Let f be holomorphic in the annulus $A = \{z \;\mid\; r \leq |z| \leq R\}$. Show that f can be extended to a holomorphic function in the disk $D(0, R) = \{z \;\mid\; |z| \leq R\}$ if and only if

$$\int_{|z|=r} f(z)\, z^n\, dz = 0, \qquad n = 0, 1, 2, \ldots.$$

   Show that if the above condition holds only for $n \geq 10$, then f is holomorphic on the disk $D(0, R)$, except maybe at 0, where it has a pole of order at most 10.